SEMI-LAGRANGIAN SCHEMES for HAMILTON JACOBI EQUATIONS

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Outline

- Optimal Control Problems (FINITE and INFINITE HORIZON)
- Dynamic Programming PRINCIPLE and HAMILTON JACOBI BELLMAN EQUATIONS (HJB)
- NUMERICAL METHODS: SEMI-LAGRANGIAN SCHEMES
- APPLICATIONS and NUMERICAL TESTS

OPTIMAL CONTROL PROBLEMS

<u>control problem</u> $\min_{\alpha(\cdot)\in\mathcal{A}} J(\alpha(\cdot))$

SET of FUNCTIONS

$\mathcal{A}$ = set of admissible controls

$\alpha(\cdot)\in\mathcal{A}=\{\alpha: [0,+\infty[\to A \mid \alpha(\cdot)$ measurable functions$\}$

A is CODOMAIN of the functions in $\mathcal{A}$

<u>controlled dynamic</u>

$$y(t)=\begin{bmatrix} y_1(t)\\ y_2(t)\\ \vdots \\ y_n(t)\end{bmatrix}\in\mathbb{R}^n$$

$$(D)\quad \begin{cases}\dot y(t)=f(y(t),\alpha(t))\\ y(0)=x\end{cases}$$

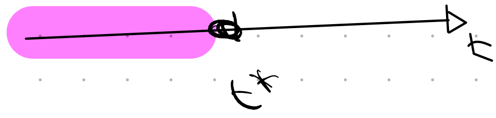
given $f:\mathbb{R}^n\times A\to\mathbb{R}^n$ DRIFT, $A\subseteq\mathbb{R}^m$ (tipically n big, m is small)

$\alpha:[0,+\infty[\to A\subseteq\mathbb{R}^m$

$$f(x,a)=\begin{bmatrix} f_1(x,a)\\ \vdots \\ f_m(x,a)\end{bmatrix}\qquad \alpha(t)=\begin{bmatrix}\alpha_1(t)\\ \vdots\\ \alpha_m(t)\end{bmatrix}$$

OSS If α is continuous w.r.t time and f is continuous with respect y, a then $f(\cdot, \alpha(\cdot))$ is continuous $\alpha(t)$

But generally α is measurable



CARATHÉODORY'S EXISTENCE THEOREM

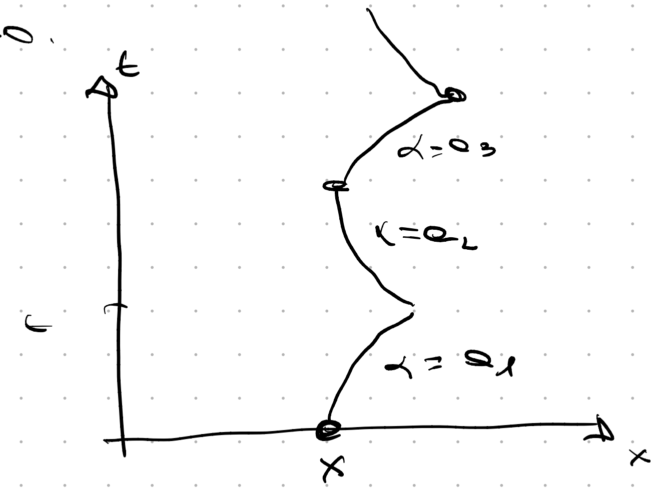
- extend solution of (P), allowing solutions that are not everywhere differentiable

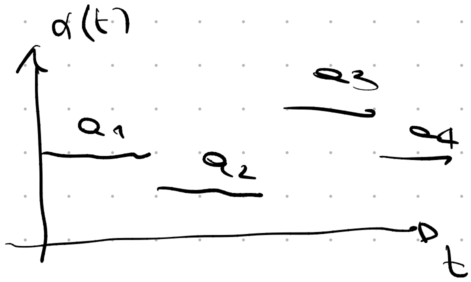
THEOREM If f is lipschitz with respect to y for each fixed t, $f(y, \alpha(\cdot))$ is measurable in t for y then $\exists!$ INTEGRAL SOLUTION of (P), i.e. $y : [0, r) \to \mathbb{R}^n$

s.t. $y(t) = x + \int_0^t f(y(s), \alpha(s))\, ds$

where $r > 0$.

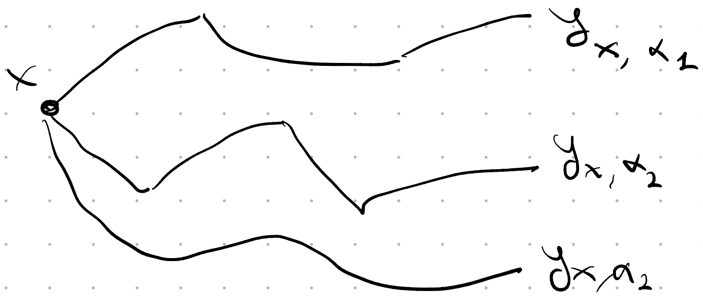
OSS





OSS If I fixed α there is only one trajectory

If I change α, I get definite trajectory



THE INFINITE HORIZON PROBLEM

Dynamics $\begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) & s > 0 \\ y(0) = x \end{cases}$ (D)

COST FUNCTION $J : \mathcal{A} \to \mathbb{R}$

set of functionals: for every $x \in \mathbb{R}^n$

$$J_x(\alpha(\cdot)) = \int_0^{+\infty} \ell(y_{x,\alpha}(s), \alpha(s)) e^{-\lambda s} \, ds$$

where $y_{x,\alpha}(\cdot)$ denotes the solution to (D)

long time horizon – INFINITE HORIZON

- $\ell : \mathbb{R}^n \times A \to \mathbb{R}$ RUNNING COST
- $\lambda \in \mathbb{R}$, DISCOUNT RATE

  $\lambda > 0$, λ big

Between all the trajectorys passing through x, I want to find the one that minimize J

Pb find α^* (if it exists) s.t.

$$J_x(\alpha^*) \leq J_x(\alpha) \quad \forall \alpha \in \mathcal{A}, \; x \in \mathbb{R}^n$$

the PONTRYAGYN MAXIMUM PRINCIPLE gives necessary condition for the optimality of α^* and y^*.

OPEN LOOP CONTROL: α^* depends on time

The system will not be able to react to possible perturbation

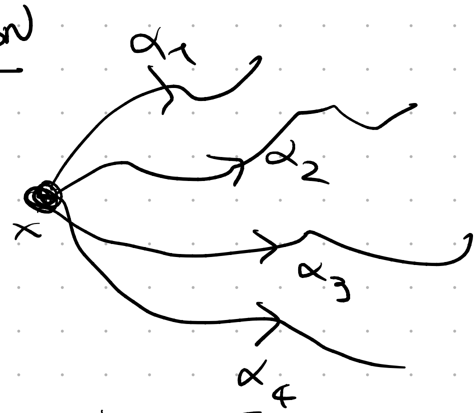
We look for a FEEDBACK CONTROL, i.e.
a control that will depend on the state
and will react to change in the theorized trajectory.
The feedback control is obtained in DYNAMIC PROGRAMMING

FEEDBACK CONTROL: controls which consider also position, i.e. a control in close form $\alpha^*(y)$

"I know how to move according where I am"

THE DYNAMIC PROGRAMMING PRINCIPLE (Bellman '60) (DPP)

Let us define the VALUE FUNCTION



$$v(x) := \inf_{\alpha \in \mathcal{A}} J_x(\alpha(\cdot))$$

x starting point for the dynamic

I choose the dynamic which minimizes the cost.

The DPP express that to achieve the minimum it is necessary

a) to evolve the system with an arbitrary control α in $[0,t]$

b) pay the cost $\int_0^t \ell(y(s), \alpha(s)) e^{-\lambda s} ds$

c) pay what remains to pay $v(y_{x,\alpha}(t)) e^{-\lambda t}$

d) minimize over all possible α

(DPP) The value function satisfies: for any $t>0$

$$v(x) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^t \ell(y_{x,\alpha}(s), \alpha(s)) e^{-\lambda s} ds + v(y_{x,\alpha}(t)) e^{-\lambda t} \right\}$$

where $y_{x,\alpha}(s)$ satisfies (E)

PROOF ('50 Richard Bellman (mainly for discrete time dynamic)

'60 become standard topic in deterministic optimal control problems

See also the book

Bardi - Capuzzo - Dolcetta "Optimal Control Problems and Viscosity Solutions of Hamilton-Jacobi-Equations"

MAIN ASSUMPTIONS

1) A is a compact subset of $\mathbb{R}^m$ (this assumption can be relaxed. It is enough A topological space)

2) f, ℓ are bounded, $\|f\|_{L^\infty} < M$, $\|\ell\|_{L^\infty} \leq M$

3) f, ℓ are continuous in (y, α)

4) f, ℓ are lip. continuous in w.r.t. the first variable, i.e.

$|f(x,a) - f(y,a)| \leq L|x-y| \quad \forall x,y \in \mathbb{R}^n, \forall a \in A$

$|\ell(x,a) - \ell(y,a)| \leq L|x-y| \quad \forall x,y \in \mathbb{R}^n, \forall a \in A$

Some Properties of the Value Function

Theorem: Under our assumptions the value function is

1) bounded
2) Lipschitz continuous on $\mathbb{R}^n$

Let us assume that there exist α^* that achieves the minimum, let us call $y^*(s) = y_{x,\alpha^*}(s)$

$$v(x) = \int_0^t \ell(y^*(s), \alpha^*(s))e^{-\lambda s}\,ds + v(y^*(t))e^{-\lambda t}$$

Add and remove $v(x)e^{-\lambda t}$ and divide by t

$$\underbrace{\frac{v(x) - v(x)e^{-\lambda t}}{t}} + \frac{v(x)e^{-\lambda t} - v(y^*(t))e^{-\lambda t}}{t} = \int_0^t \frac{\ell(y^*(s), \alpha^*(s))e^{-\lambda s}}{t}\,ds$$

$$v(x)\underbrace{\left(\frac{1 - e^{-\lambda t}}{t}\right)}_{\to \lambda} + e^{-\lambda t}\underbrace{\left(\frac{v(x) - v(y^*(t))}{t}\right)}_{\downarrow\, -\nabla v(x)\cdot f(x,\alpha^*(0))} = \underbrace{\frac{1}{t}\int_0^t \ell(y^*(s), \alpha^*(s))e^{-\lambda s}\,ds}_{\downarrow\, \ell(x,\alpha^*(0))}$$

$$\frac{1 - e^{-\lambda t}}{t} \xrightarrow{t \to 0} \lambda$$

$$-\frac{(v(y^*(t)) - v(x))}{t} \xrightarrow{t \to 0} -\nabla v(y^*(0)) \cdot \dot{y}^*(0) = -\nabla v(x) \cdot f(x, \alpha^*(0))$$

$$\frac{1}{t}\int_0^t \ell(y^*(s), \alpha^*(s))e^{-\lambda s}\,ds \xrightarrow{t \to 0} \ell(y^*(0), \alpha^*(0))$$

At the end, let us call $\alpha^*(0) = a$

$$\lambda v(x) - \nabla v(x) \cdot f(x, a) - \ell(x, a) = 0$$

STATIONARY HAMILTON-JACOBI-BELLMAN EQUATION
FIRST ORDER NON LINEAR PDE

$$(B) \qquad -\lambda v(x) + \max_{a\in A}\left[-\nabla v(x)\cdot f(x,a) - \ell(x,a)\right]=0 \quad x\in\mathbb{R}^n$$

the non linearity is due to the max

In general, the value function $v(x)$ is the solution of a first order PDE

$$H(x, v(x), \nabla v(x))=0 \quad \text{in } \mathbb{R}^n$$

H is called the Hamiltonian and it is convex in ∇v

Thm If $v(x)$ is C^1 then v satisfies (B)

(proved)

Is the viceversa true?

A priori the value function is not smooth