Équations aux dérivées partielles sur graphs

○

Partial differential equations on graphs

——

Essaouira

May 30, 2025

Julián Toledo

Departamento de Análisis Matemático

Universitat de València

Supported by Grant PID2022-136589NB-I00 funded by MCIN/AEI/10.13039/501100011 033 and FEDER and by Grant RED2022-134784-T funded by MCIN/AEI/10.13039/501 100011033.

Summary

The digital world has brought with it many different kinds of data of increasing size and complexity. Indeed, modern devices allow us to easily obtain images of higher resolution, as well as to collect data on internet searches, healthcare analytics, social networks, geographic information systems, business informatics, etc. Consequently, the study and treatment of these big data sets is of great interest and value.

Weighted discrete graphs provide a natural and flexible workspace in which to represent the data. In this context, a vertex represents a data point and each edge is weighted according to an appropriately chosen measure of "similarity" between the corresponding vertices.

Historically, the main tools for the study of graphs came from combinatorial graph theory. However, following the implementation of the graph Laplacian in the development of spectral clustering in the seventies, the theory of partial differential equations on graphs has obtained important results in this field (see, for example, [**23**], [**36**] and the references therein). This has prompted a big surge in the research of partial differential equations on graphs. Moreover, interest has been further bolstered by the study of problems in image processing. In this area of research, pixels are taken as the vertices and the “similarity” between pixels as the weights. The way in which these weights are defined depends on the problem at hand (see, for instance, [**26**] and [**35**]).

The aim of this course is to present some PDE problems in the workspace of random walk spaces, which include particularly discrete weighted graphs, and different aspects related to the operators involved in such problems. We will study:

– The total variational flow.

– The eigenvalue problem for the 1-Laplacian.

– ROF models.

Based on [39]:

Variational and Diffusion Problems in Random Walk Spaces.

José M. Mazón, Marcos Solera Diana, J. Julián Toledo-Melero.

Progress in Nonlinear Differential Equations and Their Applications, Vol. 103, Birkhäuser, 2023.

`https://link.springer.com/book/9783031335839`

Julián Toledo

Burjassot, Printemps 2025

Contents

CHAPTER 1

Random walk spaces

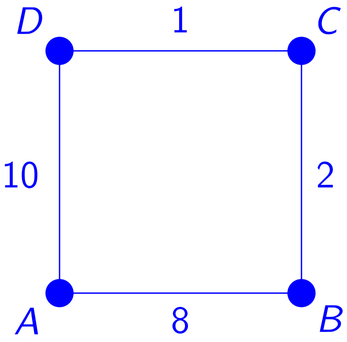


FIGURE 1. Four joined cities and their distances d (weight=1/d)

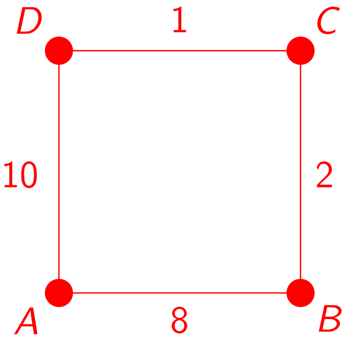


FIGURE 2. Four joined cities and their trains/day (=weights) connections

Consider a locally finite discrete graph

$$G = (V(G), E(G)),$$

where $V(G)$ is the vertex set, $E(G)$ is the edge set.

If $x, y \in V(G)$ and there is an edge connecting both vertices we write $(x, y) \in E(G)$, and also $x \sim y$.

Locally finite graph: every vertex is only contained in a finite number of edges.

– We assign to each edge $(x, y) \in E(G)$ a positive weight

$$w_{xy} = w_{yx}$$

which quantifies the connections, the relations, between vertices.

Such a graph is called a **weighted discrete graph.**

– We will also write $w_{xy} = 0$ if $(x, y) \notin E(G)$.

– There may be loops in the graph: for some $x \in V(G)$, $w_{xx} > 0$.

A finite sequence $\{x_k\}_{k=0}^{n}$ of vertices of the graph is called a *path* if $x_k \sim x_{k+1}$ for all $k = 0, 1, \ldots, n-1$.

$G = (V(G), E(G))$ is said to be *connected* if, for any two vertices $x, y \in V$, there is a path connecting x and y, that is, a path $\{x_k\}_{k=0}^{n}$ such that $x_0 = x$ and $x_n = y$.

The *length* of a path $\{x_k\}_{k=0}^n$ is defined as the number n of edges in the path.

If $G = (V(G), E(G))$ is connected, the *graph distance* $d_G(x, y)$ between any two distinct vertices x, y is defined as the minimum of the lengths of the paths connecting x and y.

This metric is independent of the weights.

It is not necessary for the most part of the course.

– For $x \in V(G)$ we define the weight (weighted degree) at x as

$$d_x := \sum_{y \sim x} w_{xy} = \sum_{y \in V(G)} w_{xy}.$$

When all the weights are 1, d_x coincides with the degree of the vertex x in a graph, that is, the number of edges containing x.

– For each $x \in V(G)$ we define the following probability measure

$$m_x^G := \frac{1}{d_x} \sum_{y \sim x} w_{xy} \, \delta_y.$$

– We also define the following measure ν_G on $V(G)$ as

$$\nu_G(A) := \sum_{x \in A} d_x, \quad A \subset V(G).$$

This is just an example of random walk space, that we will now define.

1.1. Random walk and random walk space

Let $(X, \mathcal{B})$ be a measurable space.

We may assume that the σ-algebra $\mathcal{B}$ is countably generated.

DEFINITION 1.1. A **random walk** on $(X, \mathcal{B})$ is a family of probability measures $m = (m_x)_{x \in X}$ on $\mathcal{B}$ such that $x \mapsto m_x(B)$ is a measurable function on X for each fixed $B \in \mathcal{B}$.

If m is a random walk on $(X, \mathcal{B})$ and ν is a σ-finite measure on X, then ν on X is said to be **invariant** with respect to the random walk m if

$$\nu(A) = \int_X m_x(A) d\nu(x).$$

DEFINITION 1.2. A measurable space $(X, \mathcal{B})$ together with a random walk m and an invariant measure ν with respecto to m is called a **random walk space** and denoted by $[X, \mathcal{B}, m, \nu]$.

Let us now introduce a stronger concept than invariance.

A σ-finite measure ν on X is **reversible** with respect to the random walk m if, for all $(A, B) \in \mathcal{B} \times \mathcal{B}$, we have the following symmetric property:

$$\int_A m_x(B) d\nu(x) = \int_B m_x(A) d\nu(x),$$

or, equivalently, if, for all bounded measurable function f,

$$\int_X \int_X f(x, y) dm_x(y) d\nu(x) = \int_X \int_X f(y, x) dm_x(y) d\nu(x).$$

Note that:

if ν is reversible w.r.t. $m \Rightarrow$ it is invariant w.r.t m.

DEFINITION 1.3. We say that a random walk space $[X, \mathcal{B}, m, \nu]$ is a **reversible random walk space** if ν is reversible with respect to m.

EXAMPLE 1.4. [Weighted discrete graphs] Going back to the weighted discrete graphs, if we consider σ-algebra of all subsets of $V(G)$, we have that $[V(G), \mathcal{B}, m^G, \nu_G]$ is a reversible random walk space.

EXAMPLE 1.5. [Markov chains] Let $K : X \times X \to \mathbb{R}$ be a Markov kernel on a countable space X, i.e.,

$$K(x,y) \geqslant 0 \quad \forall x, y \in X, \qquad \sum_{y \in X} K(x,y) = 1 \quad \forall x \in X.$$

Then, if

$$m_x^K(A) := \sum_{y \in A} K(x,y), \quad x \in X,\ A \subset X$$

and $\mathcal{B}$ is the σ-algebra of all subsets of X, m^K is a random walk on $(X, \mathcal{B})$.

In this ambient space, a measure π on X satisfying

$$\sum_{x \in X} \pi(x) = 1 \quad \text{and} \quad \pi(y) = \sum_{x \in X} \pi(x) K(x,y) \quad \forall y \in X,$$

is called a stationary probability measure (or steady state) on X.

Since

π is a stationary prob. measure $\Leftrightarrow$ π is and invariant prob. measure w.r..t m^K,

$[X, \mathcal{B}, m^K, \pi]$ is a random walk space.

A stationary probability measure π is reversible for K if

$$K(x, y)\pi(x) = K(y, x)\pi(y) \ \text{ for } x, y \in X.$$

Note that, given a locally finite weighted discrete graph as in Example 1.4, there is a natural definition of a Markov chain on the vertices:

$$K_G(x, y) := \frac{1}{d_x} w_{xy}.$$

We have that m^G and m^{K_G} define the same random walk.

EXAMPLE 1.6. Consider the metric measure space $(\mathbb{R}^N, d, \mathcal{L}^N)$, where d is the Euclidean distance and $\mathcal{L}^N$ the Lebesgue measure.

Let $J : \mathbb{R}^N \to [0, +\infty[$ be a measurable, nonnegative and radially symmetric function verifying $\int_{\mathbb{R}^N} J(x)dx = 1$.

Let m^J be the following random walk on $(\mathbb{R}^N, d)$:

$$m^J_x(A) := \int_A J(x-y)dy \quad \text{for } x \in \mathbb{R}^N \text{ and Borel set } A \subset \mathbb{R}^N.$$

Therefore, $[\mathbb{R}^N, d, m^J, \mathcal{L}^N]$ is a reversible (metric) random walk space.

If each individual starting at location x jumps to location y according to the probability distribution $J(x-y)$, then $m^J_x(A)$ is measuring the proportion of individuals who started at x and are arriving at A after one jump.

EXAMPLE 1.7. Given a reversible random walk space $[X, \mathcal{B}, m, \nu]$ and $\Omega \in \mathcal{B}$ with $\nu(\Omega) > 0$, let

$$m_x^{\Omega}(A) := \int_A dm_x(y) + \left(\int_{X \setminus \Omega} dm_x(y) \right) \delta_x(A) \quad \text{for } A \in \mathcal{B}_\Omega \text{ and } x \in \Omega.$$

Then, $[\Omega, \mathcal{B}_\Omega, m^{\Omega}, \nu \llcorner \Omega]$ is a reversible random walk space.

From now on we will work with **reversible random walk spaces**.

1.2. Some basic operators on random walk spaces

Given a function $f : X \to \mathbb{R}$ we define its **nonlocal gradient** $\nabla f : X \times X \to \mathbb{R}$ as

$$\nabla f(x,y) := f(y) - f(x) \quad \forall\, x, y \in X.$$

Given $\mathbf{z} : X \times X \to \mathbb{R}$ we define its ***m*-divergence** $\mathrm{div}_m \mathbf{z} : X \to \mathbb{R}$ as

$$(\mathrm{div}_m \mathbf{z})(x) := \frac{1}{2} \int_X (\mathbf{z}(x,y) - \mathbf{z}(y,x))\, dm_x(y).$$

DEFINITION 1.8. We define the *m-Laplace operator* (or *m*-***Laplacian***) as

$$\Delta_m f(x) = \int_X f(y) dm_x(y) - f(x) = \int_X (f(y) - f(x)) dm_x(y).$$

Observe that using the averaging operator $M_m f(x) = \int_X f(y) dm_x(y)$,

$$\Delta_m f = M_m f - f.$$

In the case of locally finite weighted discrete graph $G = (V, E)$ such laplacian is a normalized graph Laplacian:

$$\Delta f(x) = \frac{1}{d_x} \sum_{y \sim x} w_{xy}(f(y) - f(x)), \quad \text{for } x \in V.$$

See [**41**, Ollivier].

Observe that

$$\Delta_m f = \mathrm{div}_m(\nabla f).$$

Similarly, for $1 < p < +\infty$, we can define a m-p-Laplacian:

$$\mathrm{div}_m(|\nabla f|^{p-2}\nabla f) =: \Delta_p^m f.$$

And also, at least formally, a m-1-Laplacian:

$$\mathrm{div}_m\left(\frac{\nabla f}{|\nabla f|}\right) =: \Delta_1^m f.$$

On graphs, see also [**27**, Elmoataz, Toutain and Tenbrinck].

PROPOSITION 1.9. *(Integration by parts formula) We have that*

$$\int_X f(x)\Delta_m g(x) d\nu(x) = -\frac{1}{2}\int_X \int_X \nabla f(x,y)\nabla g(x,y) dm_x(y) d\nu(x)$$

for $f, g \in L^1(X,\nu) \cap L^2(X,\nu)$.

PROOF. By the reversibility of ν with respect to m,

$$\int_X \int_X f(x)(g(y) - g(x)) dm_x(y) d\nu(x) = \int_X \int_X f(y)(g(x) - g(y)) dm_x(y) d\nu(x).$$

Hence,

$$\int_X f(x)\Delta_m g(x) d\nu(x) = \int_X \int_X f(x)(g(y) - g(x)) dm_x(y) d\nu(x)$$

$$= \frac{1}{2}\int_X \int_X f(x)(g(y) - g(x)) dm_x(y) d\nu(x) + \frac{1}{2}\int_X \int_X f(x)(g(y) - g(x)) dm_x(y) d\nu(x)$$

$$= \frac{1}{2}\int_X \int_X f(x)(g(y) - g(x)) dm_x(y) d\nu(x) + \frac{1}{2}\int_X \int_X f(y)(g(x) - g(y)) dm_x(y) d\nu(x)$$

$$= -\frac{1}{2}\int_X \int_X \nabla f(x,y) \nabla g(x,y) dm_x(y) d\nu(x).$$

□

1.3. The nonlocal perimeter and mean curvature

For $A, B \in \mathcal{B}$, we define the **m-interaction** between A and B as

$$L_m(A, B) := \int_A \int_B dm_x(y) d\nu(x) = \int_A m_x(B) d\nu(x).$$

We have that

$$L_m(A, B) = L_m(B, A).$$

For a population which is originally distributed according to ν and which moves according to the law provided by the random walk m, $L_m(A, B)$ **measures how many individuals are moving from A to B in one jump**. The reversibility of ν with respect to m implies that this is equal to the amount of individuals moving from B to A in one jump.

DEFINITION 1.10. The m-**perimeter** of $E \in \mathcal{B}$ is defined by

$$P_m(E) := L_m(E, X \setminus E) = \int_E \int_{X\setminus E} dm_x(y) d\nu(x).$$

This notion is measuring the total flux of individuals that cross the "boundary" (in a very weak sense) of a set in one jump. So, it gives *how large* is such "boundary".

Observe that $P_m(E) = P_m(X \setminus E)$, and we have the following recognizable characterisation:

$$P_m(E) = \frac{1}{2} \int_X \int_X |\chi_E(y) - \chi_E(x)| dm_x(y) d\nu(x)$$

$$= \frac{1}{2} \int_X \int_X |\nabla \chi_E(x, y)| dm_x(y) d\nu(x).$$

Moreover, if $\nu(E) < +\infty$ then

$$P_m(E) = \nu(E) - \int_E \int_E dm_x(y) d\nu(x). \tag{1.1}$$

For the case of a weighted graph $(V(G), E(G))$. Given $A, B \subset V(G)$, one can find the following definitions:

$$\mathrm{Cut}(A, B) := \sum_{x \in A, y \in B} w_{xy},$$

and the perimeter of a set $A \subset V(G)$ as

$$|\partial A| := \mathrm{Cut}(A, A^c).$$

So, we have the same concepts.

For $[\mathbb{R}^N, d, m^J, \mathcal{L}^N]$

$$P_{m^J}(E) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\chi_E(y) - \chi_E(x)| J(x-y)\,dy\,dx,$$

coincides with the concept of J-perimeter given in **[37]** (or **[38]**),

where you can find:

If $C_J := \int_{\mathbb{R}^N} |z_N| J(z)\,dz < +\infty$, then, for $J_\epsilon(x) = \frac{1}{\epsilon^N} J\left(\frac{x}{\epsilon}\right)$, $\epsilon > 0$,

$$\lim_{\epsilon \to 0^+} C_{J_\epsilon} P_{m^{J_\epsilon}}(E) = \mathrm{Per}(E),$$

for any bounded set $E \subset \mathbb{R}^N$ of finite perimeter.

For $E \subset \Omega$,

$$\begin{aligned} P_{m^{J,\Omega}}(E) &= \frac{1}{2}\int_{\Omega}\int_{\Omega} |\chi_E(y) - \chi_E(x)| J(x-y)\,dy\,dx \\ &= P_{m^J}(E) - \int_E \left(\int_{\mathbb{R}^N \setminus \Omega} J(x-y)\,dy \right) dx. \end{aligned}$$

Exercise 1.11. If A, B, $C \in \mathcal{B}$ have pairwise ν-null intersections then

$$P_m(A \cup B \cup C) = P_m(A \cup B) + P_m(A \cup C) + P_m(B \cup C) \\ - P_m(A) - P_m(B) - P_m(C).$$

Exercise 1.12 (Submodularity). For $A,\ B \in \mathcal{B}$,

$$P_m(A \cup B) + P_m(A \cap B) \leqslant P_m(A) + P_m(B).$$

Definition 1.13. Let $E \in \mathcal{B}$. For a point $x \in X$ we define the **m-mean curvature of ∂E at x** as

$$\mathcal{H}^m_{\partial E}(x) := m_x(X \setminus E) - m_x(E).$$

Note that $\mathcal{H}^m_{\partial E}(x)$ is defined for every $x \in X$.
We have that

$$\mathcal{H}^m_{\partial E}(x) = -\mathcal{H}^m_{\partial (X \setminus E)}(x).$$

If J in Example 1.6 is continuous with compact support,

$$\lim_{\epsilon \to 0^+} C_{J_\epsilon} \mathcal{H}^{m^{J_\epsilon}}_{\partial E}(x) = (N-1) H_{\partial E}(x) \quad \text{for } x \in \partial E,$$

for any C^2-smooth set $E \subset \mathbb{R}^N$.

If $\nu(E) < +\infty$,

$$\int_E \mathcal{H}^m_{\partial E}(x) d\nu(x) = \int_E \left(1 - 2\int_E dm_x(y)\right) d\nu(x) = \nu(E) - 2\int_E \int_E dm_x(y) d\nu(x),$$

hence, having in mind (1.1), we obtain that

$$\int_E \mathcal{H}^m_{\partial E}(x) d\nu(x) = 2P_m(E) - \nu(E).$$

EXERCISE 1.14. For $[\Omega, \mathcal{B}_\Omega, m^\Omega, \nu \llcorner \Omega]$ as in Example 1.7,

$$\mathcal{H}^{m^\Omega}_{\partial E}(x) = \begin{cases} m_x(\Omega \setminus E) - m_x(E) + m_x(X \setminus \Omega) & \text{if } x \in \Omega \setminus E, \\ \\ m_x(\Omega \setminus E) - m_x(E) - m_x(X \setminus \Omega) & \text{if } x \in E. \end{cases}$$

EXERCISE 1.15. Suppose that ν is a probability measure. Then, for $D \in \mathcal{B}$, the following statements are equivalent

(i) $\Delta_m \chi_D = 0$ ν-a.e.

(ii) $P_m(D) = 0$.

(iii) $\displaystyle\frac{1}{\nu(D)} \int_D \mathcal{H}^m_{\partial D}(x) d\nu(x) = -1$.

1.4. m-Connectedness

DEFINITION 1.16. We say that $[X, \mathcal{B}, m, \nu]$ is m-connected if, for every $D \in \mathcal{B}$ with $\nu(D) > 0$ and ν-a.e. $x \in X$,

$$\sum_{n=1}^{\infty} m_x^{*n}(D) > 0.$$

The (fundamental) idea in that concept is that all parts of the space can be reached after a certain number of jumps, no matter what the starting point (except for, at most, a ν-null set of points).

The following result gives a characterization of m-connectedness in terms of the m-interaction between sets.

PROPOSITION 1.17. *The following statements are equivalent:*

(i) $[X, \mathcal{B}, m, \nu]$ *is* m*-connected.*

(ii) If $A, B \in \mathcal{B}$ *satisfy* $A \cup B = X$ *and* $L_m(A, B) = 0$*, then either* $\nu(A) = 0$ *or* $\nu(B) = 0$*.*

This result justifies the choice of the terminology used since the characterisation of m-connectedness given is in some way reminiscent of the definition of a connected topological space.

We can also speak about m-connectedness for a subset. Let $\Omega \in \mathcal{B}$ with $\nu(\Omega) > 0$, and let $\mathcal{B}_\Omega$ be the following σ-algebra

$$\mathcal{B}_\Omega := \{B \in \mathcal{B} \,:\, B \subset \Omega\},$$

we say that Ω is *m-connected (with respect to ν)* if $L_m(A, B) > 0$ for every pair of non-ν-null sets $A, B \in \mathcal{B}_\Omega$ such that $A \cup B = \Omega$. That is,

$$\Omega \text{ is } m\text{-connected} \Leftrightarrow [\Omega, \mathcal{B}_\Omega, m^\Omega, \nu \llcorner \Omega] \text{ is } m^\Omega\text{-connected.}$$

EXERCISE 1.18. Suppose that ν is a probability measure. Then, the following statements are equivalent:

(i) $[X, \mathcal{B}, m, \nu]$ is m-connected.

(ii) Δ_m is ergodic ($\Delta_m f = 0$ ν-a.e. $\Rightarrow f$ is a constant ν-a.e.).

And also, they are equivalent to:

(iii) For every $D \in \mathcal{B}$, $\Delta_m \chi_D = 0$ ν-a.e. $\Rightarrow \nu(D) = 0$ or $\nu(D) = 1$.

(iv) For every $D \in \mathcal{B}$, $0 < \nu(D) < 1 \Rightarrow P_m(D) > 0$.

(v) For every $D \in \mathcal{B}$,

$$0 < \nu(D) < 1 \;\Rightarrow\; \frac{1}{\nu(D)} \int_D \mathcal{H}^m_{\partial D}(x) d\nu(x) > -1.$$

1.5. Poincaré type inequalities

Like in the local case, Poincaré type inequalities play a very important role in this framework:

– to obtain results on the rates of convergence of the heat flow or the total variation flow for example;

– to prove existence of solutions to some pde type problems.

—— Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space, with ν a probability measure. ——

1.5.1. Global Poincaré type inequalities.

We define the (nonlocal Dirichlet) energy functional $\mathcal{H}_m : L^2(X,\nu) \to [0,+\infty)$ by

$$\mathcal{H}_m(f) := \frac{1}{4} \int_{X\times X} (f(x)-f(y))^2 dm_x(y) d\nu(x).$$

Integrating by parts (and using the reversibility of ν with respect to m),

$$\mathcal{H}_m(f) = -\frac{1}{2} \int_X f(x) \Delta_m f(x) d\nu(x).$$

We say that $[X, \mathcal{B}, m, \nu]$ satisfies a *Poincaré inequality* if there exists $\lambda > 0$ such that

$$\lambda \|f\|^2_{L^2(X,\nu)} \leqslant \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X,\nu) \text{ with } \int_X f \, d\nu = 0.$$

If we denote the mean value of $f \in L^2(X,\nu)$ (that is, the expected value of f) with respect to ν by $\nu(f)$:

$$\nu(f) := \int_X f(x)d\nu(x) = \mathbb{E}_\nu(f),$$

and its variance with respect to ν by

$$\mathrm{Var}_\nu(f) := \int_X (f(x) - \nu(f))^2 d\nu(x),$$

then $[X, \mathcal{B}, m, \nu]$ satisfies a Poincaré inequality iff

$$\lambda \mathrm{Var}_\nu(f) \leqslant \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X,\nu), \tag{1.2}$$

The spectral gap of the Laplace operator is defined as

$$\mathrm{gap}(-\Delta_m) = \inf\left\{2\mathcal{H}_m(f) : \|f\|_{L^2(X,\nu)} = 1,\ \nu(f) = 0\right\}.$$

Then, if $\mathrm{gap}(-\Delta_m) > 0$ we have that $[X, \mathcal{B}, m, \nu]$ satisfies a Poincaré inequality with $\frac{1}{2}\mathrm{gap}(-\Delta_m)$ being the best constant in such inequality.

We have that, $\mathrm{gap}(-\Delta_m) = \min \sigma(-\Delta_m)$, when $-\Delta_m$ is restricted to $H(X,\nu) := \left\{f \in L^2(X,\nu) : \nu(f) = 0\right\}$, and $\mathrm{gap}(-\Delta_m) \in [0,2]$. Then:

$$\mathrm{gap}(-\Delta_m) > 0 \Leftrightarrow 0 \notin \sigma(-\Delta_m).$$

PROPOSITION 1.19. *If $-\Delta_m$ is the sum of an invertible and a compact operator in $H(X,\nu)$, then*

$$\mathrm{gap}(-\Delta_m) > 0.$$

COROLLARY 1.20. *Consequently, if the averaging operator M_m is compact in $H(X,\nu)$ then* $\mathrm{gap}(-\Delta_m) > 0$.

If $G = (V(G), E(G))$ is a finite connected weighted discrete graph then M_{m^G} is compact and, consequently, $\mathrm{gap}(-\Delta_m^G) > 0$.

Let Ω be a bounded domain in $\mathbb{R}^N$ and let J be a kernel such that $J \in C(\mathbb{R}^N, \mathbb{R})$ is nonnegative and radially symmetric, with $J(0) > 0$ and $\int_{\mathbb{R}^N} J(x)dx = 1$. Consider the reversible metric random walk space $[\Omega, \mathcal{B}_\Omega, m^{J,\Omega}, \mathcal{L}^N]$ as defined in Example 1.7. Then, $-\Delta_{m^{J,\Omega}}$ is the sum of an invertible and a compact operator:

$$-\Delta_{m^{J,\Omega}} f(x) = \int_\Omega J(x-y)dy f(x) - \int_\Omega f(y)J(x-y)dy, \ x \in \Omega.$$

The metric random walk space associated to the locally finite weighted discrete graph G with vertex set $V(G) := \{x_3, x_4, x_5 \ldots, x_n, \ldots\}$ and weights

$$w_{x_{3n},x_{3n+1}} = \frac{1}{n^3}, \;\; w_{x_{3n+1},x_{3n+2}} = \frac{1}{n^2}, \;\; w_{x_{3n+2},x_{3n+3}} = \frac{1}{n^3},$$

for $n \geqslant 1$, and $w_{x_i,x_j} = 0$ otherwise, **does not satisfy** a Poincaré inequality.

If $\operatorname{gap}(-\Delta_m) > 0$, then Δ_m is ergodic, but the above example shows that the reverse implication does not hold in general.

PROPOSITION 1.21. *If $[X, \mathcal{B}, m, \nu]$ satisfies a Poincaré inequality, then Δ_m is ergodic (equivalently, $[X, \mathcal{B}, m, \nu]$ is m-connected).*

PROOF. Let $f \in D(\Delta_m)$ such that $\Delta_m f = 0$ ν-a.e. Then,

$$\mathcal{H}_m(f) = -\frac{1}{2}\int_X f(x)\Delta_m f(x)d\nu(x) = 0$$

and, therefore, if $[X, \mathcal{B}, m, \nu]$ satisfies a Poincaré inequality, we have that

$$\mathrm{Var}_\nu(f) = \int_X (f(x) - \nu(f))^2 d\nu(x) = 0$$

thus f is ν-a.e. a constant:

$$f(x) = \int_X f(x)d\nu(x) \quad \text{for } \nu\text{-a.e. } x \in X. \qquad \square$$

In the case of Riemannian manifolds and Markov diffusion semigroups, a usual condition required to obtain a Poincaré inequality is the positivity of the Ricci curvature of the underlying space, whose meaning is that "small balls are closer, in the 1-Wasserstein distance, than their centers are" (see [**9**, Bakry, Gentil, Ledoux], [**43**, von Reness, Sturm], [**46**, Villani]).

When the space under consideration is discrete, for instance, in the case of a graph, that concept is not as clearly applicable as in the continuous setting.

Nevertheless, in the discrete case there is a well suited concept of curvature introduced by Y. Ollivier in [**41**], coarse Ricci curvature, whose positivity ensures that a Poincaré inequality holds, in this case the balls are substitute by the measures m_x.

DEFINITION 1.22. Given random walk m on a Polish metric space (X, d) such that each measure m_x has finite first moment, for $x, y \in X$, $x \neq y$, the *Ollivier-Ricci curvature (or coarse Ricci curvature) of* $[X, d, m]$ *along* (x, y) is

$$\kappa_m(x, y) := 1 - \frac{W_1^d(m_x, m_y)}{d(x, y)}.$$

The *Ollivier-Ricci curvature of* $[X, d, m]$ is

$$\kappa_m := \inf_{\substack{x, y \in X \\ x \neq y}} \kappa_m(x, y).$$

m_x has finite first moment if for some (then, for any) $y_0 \in X$, we have that $\int_X d(y, y_0) dm_x(y) < +\infty$.

$W_1^d(m_x, m_y)$ is the 1-*Wasserstein distance* between m_x and m_y.

THEOREM 1.23. [**41**, Ollivier] *Let $[X, d, m, \nu]$ be a reversible metric random walk space, with $[X, d, m]$ as in the above definition (ν a probability measure). Suppose that*

$$\sigma := \int_X \int_X \int_X d(y,z)^2 dm_x(y) dm_x(z) d\nu(x) < +\infty.$$

If the Ollivier-Ricci curvature of $[X, d, m]$, κ_m, is positive, then

$$\kappa_m \leqslant \operatorname{gap}(-\Delta_m).$$

CHAPTER 2

The total variation flow in random walk spaces

The total variation flow has remained one of the most popular tools in Image Processing since its introduction as a means of solving the denoising problem by Rudin, Osher and Fatemi [**44**].

But also for nonlocal models with neighbourhood filters ([**15**, Buades, Coll, Morel].

And for models on weighted graphs ([**26**, Elmoataz, Lezoray, Bougleux], [**35**, Lozes, Elmoataz, Lézoray]).

Therefore, the study of the 1-Laplacian operator and the total variation flow in random walk spaces has a potentially broad scope of application.

—— Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space.

2.1. The m-total variation

We define the space of nonlocal bounded variation functions as:

$$BV_m(X,\nu) = \left\{ u : X \to \mathbb{R} \ \text{ measurable} : \int_X \int_X |u(y) - u(x)| dm_x(y) d\nu(x) < \infty \right\}.$$

Moreover, the ***m*-total variation** of a function $u \in BV_m(X,\nu)$ is defined by

$$TV_m(u) = \frac{1}{2} \int_X \int_X |u(y) - u(x)| dm_x(y) d\nu(x).$$

REMARK 2.1. With this definition, we have that

$$P_m(E) = TV_m(\chi_E).$$

The space $BV_m(X,\nu)$ could be seen as the *nonlocal* counterpart of classical local bounded variation spaces (BV-spaces). However, although they represent analogous concepts in different settings, the local classical BV-spaces and the nonlocal BV-spaces are of a different nature. For example, $L^1(X,\nu) \subset BV_m(X,\nu)$ (and $TV_m(u) \leqslant \|u\|_{L^1(X,\nu)}$) in contrast with classical local bounded variation spaces that are, by definition, contained in L^1.

For the random walk space associated to a weighted graph,

$$TV_{m^G}(u) = \frac{1}{2} \sum_{x \in V(G)} \sum_{y \sim x} w_{xy} |u(y) - u(x)|,$$

which coincides with the anisotropic total variation defined in [**31**, van Gennip, Guillen, Osting, Bertozzi].

We have a coarea formula relating the m-total variation of a function u with the m-perimeter of its superlevel sets,

$$E_t(u) := \{x \in X : u(x) > t\}.$$

THEOREM 2.2 (**Coarea formula**). *For $u \in BV_m(X, \nu)$,*

$$TV_m(u) = \int_{-\infty}^{+\infty} P_m(E_t(u))\, dt.$$

PROOF. Let $u \in BV_m(X,\nu)$. Since

$$u(x) = \int_0^{+\infty} \chi_{E_t(u)}(x)\,dt - \int_{-\infty}^0 (1 - \chi_{E_t(u)}(x))\,dt \quad \forall x \in X,$$

we have

$$u(y) - u(x) = \int_{-\infty}^{+\infty} \chi_{E_t(u)}(y) - \chi_{E_t(u)}(x)\,dt \quad \forall x, y \in X.$$

Moreover, since $u(y) \geqslant u(x)$ implies $\chi_{E_t(u)}(y) \geqslant \chi_{E_t(u)}(x)$, we obtain that

$$|u(y) - u(x)| = \int_{-\infty}^{+\infty} |\chi_{E_t(u)}(y) - \chi_{E_t(u)}(x)|\,dt.$$

Therefore, we get (using Tonelli-Hobson's Theorem)

$$\begin{aligned} TV_m(u) &= \frac{1}{2}\int_X \int_X |u(y) - u(x)| dm_x(y) d\nu(x) \\ &= \frac{1}{2}\int_X \int_X \left(\int_{-\infty}^{+\infty} |\chi_{E_t(u)}(y) - \chi_{E_t(u)}(x)| dt \right) dm_x(y) d\nu(x) \\ &= \int_{-\infty}^{+\infty} \left(\frac{1}{2}\int_X \int_X |\chi_{E_t(u)}(y) - \chi_{E_t(u)}(x)| dm_x(y) d\nu(x) \right) dt \\ &= \int_{-\infty}^{+\infty} P_m(E_t(u)) dt. \end{aligned}$$

$\square$

Set $X_m^2(X,\nu) := \left\{ \mathbf{z} \in L^\infty(X \times X, \nu \otimes m_x) : \mathrm{div}_m \mathbf{z} \in L^2(X,\nu) \right\}$.

We can characterize the m-total variation and the m-perimeter using the m-divergence operator as in the local case ([**5**, Ambrosio, Fusco, Pallara]).

PROPOSITION 2.3. *For $u \in BV_m(X,\nu) \cap L^2(X,\nu)$, we have*

$$TV_m(u) = \sup \left\{ \int_X u(x)(\mathrm{div}_m \mathbf{z})(x) d\nu(x) \ : \ \mathbf{z} \in X_m^2(X,\nu), \ \|\mathbf{z}\|_{L^\infty(X\times X, \nu\otimes m_x)} \leqslant 1 \right\}.$$

In particular, for any $E \in \mathcal{B}$ with $\nu(E) < \infty$, we have

$$P_m(E) = \sup \left\{ \int_E (\mathrm{div}_m \mathbf{z})(x) d\nu(x) \ : \ \mathbf{z} \in X_m^2(X,\nu), \ \|\mathbf{z}\|_{L^\infty(X\times X, \nu\otimes m_x)} \leqslant 1 \right\}.$$

Green formula:

$$\int_X u(x)(\mathrm{div}_m \mathbf{z})(x) d\nu(x) = -\frac{1}{2} \int_X \int_X \nabla u(x,y) \mathbf{z}(x,y) dm_x(y) d\nu(x).$$

2.2. The m-1-Laplacian

DEFINITION 2.4. We define in $L^2(X,\nu)\times L^2(X,\nu)$ the multivalued operator Δ_1^m by: $(u,v)\in\Delta_1^m$ if there exists there exists $\mathbf{g}\in L^\infty(X\times X,\nu\otimes m_x)$ antisymmetric with $\|\mathbf{g}\|_{L^\infty(X\times X,\nu\otimes m_x)}\leqslant 1$ such that

$$v(x)=\int_X \mathbf{g}(x,y)\,dm_x(y)\quad \text{for } \nu\text{-a.e } x\in X,$$

and

$$\mathbf{g}(x,y)\in \operatorname{sign}(u(y)-u(x))\quad \text{for } (\nu\otimes m_x)\text{-a.e. } (x,y)\in X\times X.$$

For weighted finite graphs (Chang in [**19**] and Hein and Bühler in [**32**]):
$(u,v)\in\Delta_1^{m^G}$ if $\exists\mathbf{g}\in L^\infty(V(G)\times V(G))$ antisymmetric such that $\|\mathbf{g}\|_{L^\infty(V(G)\times V(G))}\leqslant 1$,

$$v(x)=\frac{1}{d_x}\sum_{y\in V(G)}\mathbf{g}(x,y)w_{xy}\quad \forall\, x\in V(G),$$

and

$$\mathbf{g}(x,y)\in \operatorname{sign}(u(y)-u(x))\quad \text{for } (x,y)\in V(G)\times V(G).$$

We have that $v \in \Delta_1^m(u)$ iff
(1) there exists $\mathbf{z} \in X_m^2(X,\nu)$ with $\|\mathbf{z}\|_{L^\infty(X\times X,\nu\otimes m_x)} \leqslant 1$ such that

$$v = \mathrm{div}_m \mathbf{z}$$

and one of the two following properties:

$$-\int_X u(x)v(x)d\nu(x) = TV_m(u);$$

or

$$\frac{1}{2}\int_X\int_X \nabla u(x,y)\mathbf{z}(x,y)dm_x(y)d\nu(x) = TV_m(u);$$

iff
(2) there exists $\mathbf{g} \in L^\infty(X\times X, \nu\otimes m_x)$ antisymmetric with $\|\mathbf{g}\|_\infty \leqslant 1$ such that

$$v(x) = \int_X \mathbf{g}(x,y)\,dm_x(y) \quad \text{for } \nu\text{-a.e } x\in X,$$

$$-\int_X\int_X \mathbf{g}(x,y)dm_x(y)\,u(x)d\nu(x) = TV_m(u).$$

THEOREM 2.5. *$-\Delta_1^m$ is maximal monotone with dense domain. In fact, it an the m-completely accretive operator.*

$-\Delta_1^m$ is the subdifferential of the m-total variation.

Hence, for the Cauchy problem

(2.1) $$\begin{cases} u_t - \Delta_1^m u \ni 0 & \text{in } (0,T)\times X, \\ u(0,x) = u_0(x) & x \in X, \end{cases}$$

which equation rewrites the formal nonlocal equation

$$u_t(x,t) = \int_X \frac{u(y,t)-u(x,t)}{|u(y,t)-u(x,t)|} dm_x(y), \quad x \in X, t \geqslant 0,$$

we have:

THEOREM 2.6. *For every $u_0 \in L^2(X,\nu)$ and any $T > 0$, there exists a unique solution of the Cauchy Problem (2.1) in $(0,T)$ in the following sense: $u \in W^{1,1}(0,T;L^2(X,\nu))$, $u(0,\cdot) = u_0$ in $L^2(X,\nu)$, and, for almost all $t \in (0,T)$,*

$$u_t(t,\cdot) - \Delta_1^m u(t) \ni 0.$$

Moreover, we have the following contraction and maximum principle ($1 \leqslant q \leqslant +\infty$):

$$\|(u(t)-v(t))^+\|_{L^q(X,\nu)} \leqslant \|(u_0-v_0)^+\|_{L^q(X,\nu)} \quad \forall\, 0 < t < T,$$

for any pair of solutions u and v of problem (2.1) with initial datum u_0 and v_0, respectively.

Given $u_0 \in L^2(X, \nu)$, we denote the unique solution of Problem (2.1) by $e^{t\Delta_1^m} u_0$.

We call the semigroup $\{e^{t\Delta_1^m}\}_{t \geqslant 0}$ in $L^2(X, \nu)$ the *total variational flow* in $[X, \mathcal{B}, m, \nu]$ or the *m-total variational flow*.

—— Let $[X,\mathcal{B},m,\nu]$ be a reversible random walk space, m-connected, and ν is a probability measure. ——

2.3. Asymptotic behaviour

THEOREM 2.7. *For $u_0 \in L^2(X,\nu)$,*

$$\int_X e^{t\Delta_1^m} u_0 d\nu = \int_X u_0 d\nu \quad \textit{for every } t \geqslant 0.$$

THEOREM 2.8. *For every $u_0 \in L^2(X,\nu)$,*

$$\lim_{t\to\infty} e^{t\Delta_1^m} u_0 = \int_X u_0 d\nu.$$

We can specify a rate of convergence of the total variational flow $(e^{t\Delta_1^m})_{t\geqslant 0}$ when a *Poincaré type inequality* holds.

If $[X,\mathcal{B},m,\nu]$ satisfies a $(p,1)$-Poincaré inequality, set

$$\lambda_{m,p} := \inf\left\{ \frac{TV_m(u)}{\|u\|_{L^p(X,\nu)}} \,:\, \|u\|_{L^p(X,\nu)} \neq 0,\ \nu(u)=0 \right\},$$

to the best constant in such inequality.

THEOREM 2.9. *If $[X,\mathcal{B},m,\nu]$ satisfies a $(1,1)$-Poincaré inequality, then, for any $u_0\in L^2(X,\nu)$,*

$$\left\| e^{t\Delta_1^m}u_0 - \nu(u_0)\right\|_{L^1(X,\nu)} \leqslant \frac{1}{2\lambda_{m,1}} \frac{\|u_0\|^2_{L^2(X,\nu)}}{t} \quad \forall t>0.$$

When $[X,\mathcal{B},m,\nu]$ satisfies a $(2,1)$-Poincaré inequality, the solution of the total variational flow reaches the steady state in finite time.

THEOREM 2.10. *Suppose that* $[X, d, m, \nu]$ *satisfies a* $(2,1)$*-Poincaré inequality. Then, for any* $u_0 \in L^2(X,\nu)$,

$$\|e^{t\Delta_1^m} u_0 - \nu(u_0)\|_{L^2(X,\nu)} \leqslant \left(\|u_0 - \nu(u_0)\|_{L^2(X,\nu)} - \lambda_{m,2} t \right)^+ \quad \forall t \geqslant 0.$$

For the **extinction time**,

$$T^*(u_0) := \inf \left\{ t > 0 \ : \ e^{t\Delta_1^m} u_0 = \nu(u_0) \right\},$$

we have that

$$\|u_0 - \nu(u_0)\|_{m,*} \leqslant T^*(u_0) \leqslant \frac{1}{\lambda_{m,2}} \|u_0 - \nu(u_0)\|_{L^2(X,\nu)},$$

where $\|f\|_{m,*} := \inf \left\{ \|\mathbf{z}\|_{L^\infty(X\times X, \nu\otimes m_x)} : f = \mathrm{div}_m(\mathbf{z}) \right\}.$

CHAPTER 3

The eigenvalue problem for the m-1-Laplacian

Further motivation for the study of the 1-Laplacian operator comes from spectral clustering. Partitioning data into sensible groups is a fundamental problem in machine learning, computer science, statistics and science in general. In these fields, it is usual to face large amounts of empirical data, and getting a first impression of these data by identifying groups with similar properties has proved to be very useful.

One of the most popular approaches to this problem is to find the best balanced cut of a graph representing the data, such as the Cheeger ratio cut ([**22**, Cheeger]) which we will now introduce.

Consider a finite weighted connected graph $G = (V, E)$, where $V = \{x_1, \ldots, x_n\}$ is the set of vertices (or nodes) and E the set of edges, which are weighted by a function $w_{ji} = w_{ij} \geqslant 0$, $(x_i, x_j) \in E$. In this context, the Cheeger cut value of a partition $\{S, S^c\}$ ($S^c := V \setminus S$) of V is defined as

$$\mathcal{C}(S) := \frac{\mathrm{Cut}(S, S^c)}{\min\{\mathrm{vol}(S), \mathrm{vol}(S^c)\}},$$

where $\mathrm{Cut}(A, B) = \sum_{x_i \in A, x_j \in B} w_{ij}$ and $\mathrm{vol}(S)$ is the volume of S, defined as $\mathrm{vol}(S) := \sum_{x_i \in S} d_{x_i}$, being d_{x_i} the weight at the vertex x_i .

The minimum of such values,

$$h(G) := \min_{S \subset V} \mathcal{C}(S),$$

is called the *Cheeger constant*, and a partition $\{S, S^c\}$ of V is called a *Cheeger cut* of G if $h(G) = \mathcal{C}(S)$.

The Cheeger minimization problem of computing $h(G)$ is NP-hard ([**32**, Hein, Bühler], [**45**, Szlam, Bresson]).

However, $h(G)$ can be approximated by the first positive eigenvalue λ_1 of the $-\Delta_m$ thanks to the following Cheeger inequality ([**23**, Chung]):

$$\frac{\lambda_1}{2} \leqslant h(G) \leqslant \sqrt{2\lambda_1}.$$

The nonlocal version of the classical Cheeger inequality.

This motivates the spectral clustering method (see for example [**36**, von Luxburg]), which, in its simplest form, thresholds the first positive eigenvalue of the $-\Delta_m$ to get an approximation to the Cheeger constant and to a Cheeger cut.

If u is an eigenfunction with eigenvalue $\lambda_2(G)$, then

$$\Big\{ \{x \in V \ : \ u(x) \geqslant 0\}, \ \{x \in V \ : \ u(x) < 0\} \Big\}$$

approximates a Cheeger cut of G.

In order to achieve a better approximation than the one provided by the classical spectral clustering method, a spectral clustering based on the graph p-Laplacian was developed in [**16**, Bühler, Hein], where it is showed that the second eigenvalue of the graph p-Laplacian tends to the Cheeger constant $h(G)$ as $p \to 1^+$.

In [**45**, Szlam, Bresson] the idea was further developed by directly considering the variational characterization of the Cheeger constant $h(G)$

$$h_m(G) = \min_{u \in L^1} \frac{TV_{m^G}(u)}{\|u - \mathrm{median}(u))\|_1},$$

where (as defined above)

$$TV_{m^G}(u) := \frac{1}{2} \sum_{i,j=1}^{n} w_{ij} |u(x_i) - u(x_j)|.$$

The subdifferential of the energy functional TV_m is $-\Delta_{m^G}$.

Using the nonlinear eigenvalue problem

$$\lambda\, \mathrm{sign}(u) \in -\Delta_1 u,$$

the theory of 1-Spectral Clustering is developed by Chang, Shao an Zhang in [**19**], [**20**], [**21**] and Hein and Bühler in [**32**].

3.1. m-Cheeger and m-calibrable Sets

—— Assume that $[X, \mathcal{B}, m, \nu]$ is m-connected. ——

Given $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$, the *m-Cheeger constant* of Ω is defined as

$$h_1^m(\Omega) := \inf \left\{ \frac{P_m(E)}{\nu(E)} \ : \ E \in \mathcal{B}_\Omega, \ \nu(E) > 0 \right\}. \tag{3.1}$$

If $E \in \mathcal{B}_\Omega$ minimizes (3.1), then E is said to be an *m-Cheeger set* of Ω.

Ω is said *m-calibrable* it is an m-Cheeger set of itself, that is, if

$$h_1^m(\Omega) = \frac{P_m(\Omega)}{\nu(\Omega)}.$$

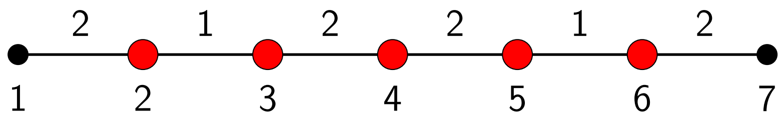
Note that, by (1.1), we have that $h_1^m(\Omega) \leqslant 1$.

Notation: given $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$, we will denote

$$\lambda_\Omega^m := \frac{P_m(\Omega)}{\nu(\Omega)}.$$

EXERCISE 3.1. Consider the metric random walk space associated to a locally finite weighted discrete graph $G = (V(G), E(G))$ having more than two vertices and no loops (i.e., $w_{xx} = 0$ for all $x \in V$). Then, any subset consisting of two vertices is m^G-calibrable.

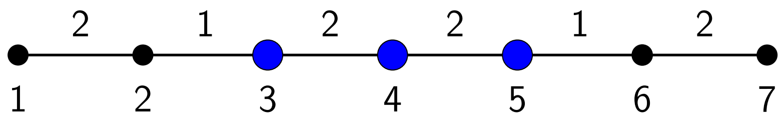
EXAMPLE 3.2. Let $G = (V(G), E(G))$ be the finite weighted discrete graph:



If $E_1 = B(4, \frac{5}{2}) = \{2, 3, \ldots, 6\}$,

$$\frac{P_{m^G}(E_1)}{\nu_G(E_1)} = \frac{w_{12} + w_{67}}{d_2 + d_3 + d_4 + d_5 + d_6} = \frac{1}{4}.$$

However, taking $E_2 = B(4, \frac{3}{2}) = \{3, 4, 5\} \subset E_1$,

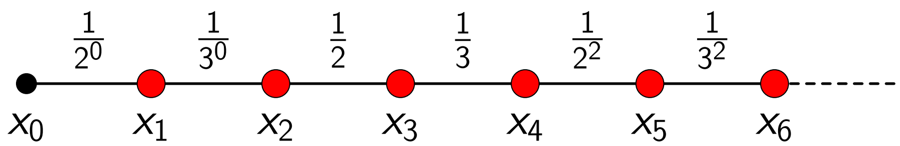


we have

$$\frac{P_{m^G}(E_2)}{\nu_G(E_2)} = \frac{w_{23} + w_{56}}{d_3 + d_4 + d_5} = \frac{1}{5}.$$

Consequently, the ball $B(4, \frac{5}{2})$ is not m^G-calibrable.

EXAMPLE 3.3. Let $G = (V(G), E(G))$ be the finite weighted discrete graph, where $V(G) = \{x_0, x_1, \ldots, x_n \ldots\}$:



If $\Omega := \{x_1, x_2, x_3 \ldots\}$, then $\dfrac{P_{m^G}(D)}{\nu_G(D)} > 0$ for every $D \subset \Omega$ with $\nu_G(D) > 0$ but $h_1^m(\Omega) = 0$. Therefore, Ω has no m-Cheeger set.

For $[\mathbb{R}^N, d, m^J, \mathcal{L}^N]$, the concepts of m-Cheeger set and m-calibrable set coincide with the concepts of J-Cheeger set and J-calibrable set introduced in [**37**] (see also [**38**]), where it is shown that each ball is a J-calibrable set.

It is well known (see [**28**, Fridman, Kawoh]) that, for a bounded smooth domain $\Omega \subset \mathbb{R}^N$, the classical Cheeger constant

$$h_1(\Omega) := \inf \left\{ \frac{Per(E)}{|E|} \ : \ E \subset \Omega, \ |E| > 0 \right\},$$

is an optimal Poincaré constant:

$$h_1(\Omega) = \inf \left\{ \frac{\displaystyle\int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1}}{\|u\|_{L^1(\Omega)}} \ : \ u \in BV(\Omega), \ \|u\|_{L^\infty(\Omega)} = 1 \right\}.$$

A nonlocal version of this result is:

THEOREM 3.4. *Let $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$. Then,*

$$h_1^m(\Omega) = \inf \left\{ \frac{TV_m(u)}{\displaystyle\int_X u(x) d\nu(x)} : u \in L^1(X,\nu)\setminus\{0\}, \ u = 0 \text{ in } X\setminus\Omega, \ u \geqslant 0 \right\}.$$

PROOF. Given $E \in \mathcal{B}$ with $\nu(E) > 0$, we have

$$\frac{TV_m(\chi_E)}{\|\chi_E\|_{L^1(X,\nu)}} = \frac{P_m(E)}{\nu(E)}.$$

Therefore, $\inf\{\ldots\} \leqslant h_1^m(\Omega)$. For the opposite inequality we will follow an idea used in [**28**]. Given $u \in L^1(X,\nu)\setminus\{0\}$, with $u = 0$ in $X\setminus\Omega$ and $u \geqslant 0$, we have

$$\begin{aligned} TV_m(u) &= \int_0^{+\infty} P_m(E_t(u))\, dt = \int_0^{\|u\|_{L^\infty(X,\nu)}} \frac{P_m(E_t(u))}{\nu(E_t(u))} \nu(E_t(u))\, dt \\ &\geqslant h_1^m(\Omega) \int_0^{+\infty} \nu(E_t(u))\, dt = h_1^m(\Omega) \int_X u(x) d\nu(x) \end{aligned}$$

where the first equality follows by the coarea formula (Theorem 2.2) and the last one by Cavalieri's Principle. Taking the infimum over u in the above expression we get $\inf\{\ldots\} \geqslant h_1^m(\Omega)$. $\square$

THEOREM 3.5. [**4**, Alter, Caselles, Chambolle] *Given a bounded convex set Ω of $\mathbb{R}^N$ of class $C^{1,1}$ ($|\Omega| > 0$), the following assertions are equivalent:*

(a) $\dfrac{Per(\Omega)}{|\Omega|} = \inf \left\{ \dfrac{Per(E)}{|E|} : E \subset \Omega, |E| > 0, Per(E) < \infty \right\}.$

(b) χ_Ω *satisfies* $-\Delta_1 \chi_\Omega = \lambda \chi_\Omega$, *where* $\Delta_1 u := \operatorname{div}\left(\frac{Du}{|Du|}\right)$,
(observe that necessarily $\lambda = \dfrac{Per(\Omega)}{|\Omega|}$*).*

(c) $(N-1) \operatorname*{ess\,sup}_{x \in \partial\Omega} \mathcal{H}_{\partial\Omega}(x) \leqslant \dfrac{Per(\Omega)}{|\Omega|}.$

In the following results, we will see that the nonlocal counterparts of some of the implications in this theorem also hold true in our setting, while others do not.

The next result is the nonlocal version of the fact that (a) is equivalent to (b) in Theorem 3.5.

THEOREM 3.6. *Let $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$. Then, the following assertions are equivalent:*

(i) Ω is m-calibrable,

(ii) $\exists \lambda > 0$ and a measurable function $\tau : X \to \mathbb{R}$ equal to 1 in Ω such that

$$-\lambda \tau \in \Delta_1^m \chi_\Omega \quad \text{in } X,$$

(iii)

$$-\lambda_\Omega^m \tau^* \in \Delta_1^m \chi_\Omega \quad \text{in } X,$$

for $\tau^*(x) = \chi_\Omega(x) - \frac{1}{\lambda_\Omega^m} m_x(\Omega) \chi_{X \setminus \Omega}(x)$.

That is,

$$\Omega \text{ is } m\text{-calibrable} \Leftrightarrow -\lambda_\Omega^m \chi_\Omega + m_{(.)}(\Omega)\,\chi_{X\setminus\Omega} \in \Delta_1^m \chi_\Omega .$$

Let $\nu(X) < \infty$. For $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$,

the equation

$$-\lambda_\Omega^m \chi_\Omega \in \Delta_1^m \chi_\Omega \quad \text{in } X$$

does not hold true.

However, if $\nu(X) = +\infty$, it may be satisfied:

Consider the metric random walk space $[\mathbb{R}, d, m^J, \mathcal{L}^1]$ with $J = \frac{1}{2}\chi_{[-1,1]}$. Then,

$$-\lambda_{[-1,1]}^{m^J} \chi_{[-1,1]} \in \Delta_1^{m^J} \chi_{[-1,1]}, \qquad \lambda_{[-1,1]}^{m^J} = \frac{1}{4}.$$

As a consequence of Theorem 3.5, it holds that
a bounded convex set $\Omega \subset \mathbb{R}^N$ is calibrable if, and only if,
$u(t,x) = \left(1 - \frac{\text{Per}(\Omega)}{|\Omega|}t\right)^+ \chi_\Omega(x)$ is a solution of the Cauchy problem

$$\begin{cases} u_t - \Delta_1 u \ni 0 & \text{in } (0,\infty) \times \mathbb{R}^N, \\ u(0) = \chi_\Omega. \end{cases}$$

That means, a calibrable convex set Ω is that for which the gradient descent flow associated to the total variation tends to decrease linearly the height of χ_Ω without distortion of its boundary.

We can obtain a similar result in our context if we introduce an absorption term in the corresponding Cauchy problem.

The appearance of this term is due to the nonlocality of the diffusion considered.

Let $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$. Ω is m-calibrable if, and only if,

$u(t)(x) = \left(1 - \lambda_\Omega^m t\right)^+ \chi_\Omega(x)$ is a solution of

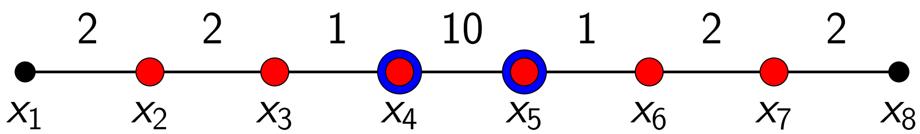
$$\begin{cases} u_t(t)(x) - \Delta_1^m u(t)(x) \ni -m_x(\Omega)\, \chi_{X\setminus\Omega}(x)\chi_{[0,1/\lambda_\Omega^m)}(t) & (t,x) \in (0,\infty) \times X, \\ u(0)(x) = \chi_\Omega(x), & x \in X. \end{cases}$$

The following result relates the m-calibrability of a set with its m-mean curvature. This is the nonlocal version of one of the implications in the equivalence between (a) and (c) in Theorem 3.5.

PROPOSITION 3.7. *Let* $\Omega \in \mathcal{B}$ *with* $0 < \nu(\Omega) < \nu(X)$. *Then,*

$$\Omega \ \ m\text{-}calibrable \ \Rightarrow \ \ \nu\text{-}\operatorname*{ess\,sup}_{x\in\Omega} \mathcal{H}^m_{\partial\Omega}(x) \leqslant \frac{P_m(\Omega)}{\nu(\Omega)}.$$

The converse of Proposition 3.7 is not true in general:



For $\Omega = \{x_2, x_3, \ldots, x_7\}$, $E = \{x_4, x_5\}$, we have

$$\mathcal{H}^m_{\partial\Omega}(x) \leqslant 0 \ \ \forall x \in \Omega, \quad \lambda^m_\Omega = \frac{1}{9}, \quad \lambda^m_E = \frac{1}{11}.$$

3.2. Eigenvalues of $-\Delta_1^m$

In this section we introduce the eigenvalues of the operator $-\Delta_1^m$ and its relation with the Cheeger minimization problem. For the particular case of finite weighted discrete graphs where the weights are either 0 or 1, this problem was first studied by Hein and Bühler ([**32**]) and a more complete study was subsequently performed by Chang in [**19**] (see also [**20**], [**21**]).

DEFINITION 3.8. A pair $(\lambda, u) \in \mathbb{R} \times L^2(X, \nu)$ is called an *m-eigenpair* of the operator $-\Delta_1^m$, on X if $\|u\|_{L^1(X,\nu)} = 1$ and there exists $\xi \in \mathrm{sign}(u)$ (i.e., $\xi(x) \in \mathrm{sign}(u(x))$ for every $x \in X$) such that

$$\lambda \xi \in \partial \mathcal{F}_m(u) = -\Delta_1^m u.$$

The function u is called an *m-eigenfunction of* $-\Delta_1^m$ and λ an *m-eigenvalue of* $-\Delta_1^m$ associated to u.

Observe that, if (λ, u) is an m-eigenpair of $-\Delta_1^m$, then $(\lambda, -u)$ is also an m-eigenpair of $-\Delta_1^m$.

REMARK 3.9. Let $(\lambda, u) \in \mathbb{R} \times L^2(X,\nu)$, $\|u\|_{L^1(X,\nu)} = 1$. We have:

$$(\lambda, u) \text{ is an } m\text{-eigenpair of } -\Delta_1^m$$
$$\Updownarrow$$

there exists $\xi \in \operatorname{sign}(u)$, and there exists $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ antisymmetric with $\|\mathbf{g}\|_\infty \leqslant 1$, such that

$$-\int_X \mathbf{g}(x,y)\, dm_x(y) = \lambda \xi(x) \quad \text{for } \nu\text{-a.e. } x \in X,$$

and one of this three properties:

$$-\int_X \int_X \mathbf{g}(x,y) dm_x(y)\, u(x) d\nu(x) = TV_m(u),$$

or

$$\mathbf{g}(x,y)(u(y)-u(x)) = |u(y)-u(x)| \quad \text{for } \nu \otimes m_x\text{-a.e. } (x,y) \in X \times X,$$

or

$$\lambda = TV_m(u).$$

REMARK 3.10. Note that, since $TV_m(u) = \lambda$ for any m-eigenpair (λ, u) of $-\Delta_1^m$, then

$$\begin{aligned}\lambda &= TV_m(u) = \frac{1}{2}\int_X \int_X |u(y) - u(x)| dm_x(y) d\nu(x) \\ &\leqslant \frac{1}{2}\int_X \int_X (|u(y)| + |u(x)|) dm_x(y) d\nu(x) = \|u\|_1 = 1,\end{aligned}$$

thus

$$0 \leqslant \lambda \leqslant 1.$$

Observe that, if a locally finite weighted discrete graph contains a vertex x with no loop, i.e. $w_{x,x} = 0$, then $\left(1, \frac{1}{d_x}\delta_x\right)$ is an m-eigenpair of $-\Delta_1^{m^G}$. Conversely, if 1 is an m-eigenvalue of $-\Delta_1^{m^G}$, then there exists at least one vertex in the graph with no loop (this is an exercise after Proposition 3.19).

THEOREM 3.11. *Assume that* $[X, \mathcal{B}, m, \nu]$ *is* m*-connected. Let* $\Omega \in \mathcal{B}$ *with* $0 < \nu(\Omega) < \nu(X)$*. We have:*

(i) If $(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega)$ *is an* m*-eigenpair of* $-\Delta_1^m$*, then* Ω *is* m*-calibrable.*

(ii) If Ω *is* m*-calibrable and*

$$m_x(\Omega) \leqslant \lambda_\Omega^m \quad \text{for } \nu\text{-a.e. } x \in X \setminus \Omega, \tag{3.2}$$

then $(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega)$ *is an* m*-eigenpair of* $-\Delta_1^m$*.*

The reverse implications in (i) and (ii) are false in general.

PROOF. (i): Since $(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega)$ is an m-eigenpair of $-\Delta_1^m$, there exists $\xi \in \mathrm{sign}(\chi_\Omega)$ such that $-\lambda_\Omega^m \xi \in \Delta_1^m(\chi_\Omega)$. Then, by Theorem 3.6, we have that Ω is m-calibrable.

(ii): If Ω is m-calibrable, by Theorem 3.6, we have

$$-\lambda_\Omega^m \tau^* \in \Delta_1^m \chi_\Omega \quad \text{in } X$$

for

$$\tau^*(x) = \begin{cases} 1 & \text{if } x \in \Omega, \\ -\dfrac{1}{\lambda_\Omega^m} m_x(\Omega) & \text{if } x \in X \setminus \Omega. \end{cases}$$

Now, by (3.2), we have that $\tau^* \in \mathrm{sign}(\chi_\Omega)$ and, consequently, $\left(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega\right)$ is an m-eigenpair of $-\Delta_1^m$. $\square$

3.3. The m-Cheeger constant

In 1969, Jeff Cheeger [**22**] (see also Polya and Szego [**42**]) proved his famous inequality

$$\frac{h_M^2}{2} \leqslant \lambda_1(\Delta_M),$$

where M is a compact manifold, $\lambda_1(\Delta_M)$ is the first non-trivial eigenvalue of the Laplace Beltrami operator Δ_M on $L^2(M, \mathrm{vol})$ and the Cheeger constant h_M is defined as follows:

$$h_M := \inf \frac{\mathrm{Area}(\partial S)}{\min(\mathrm{vol}(S), \mathrm{vol}(M \setminus S))},$$

where the infimum runs over all $S \subset M$ with sufficiently smooth boundary.

On graphs, the first results regarding Cheeger's bound for the first positive eigenvalue of the graph Laplacian are due to Dodziuk [**24**] and Alon and Milmann [**3**].

—— Assume that ν is a probability measure. ——

DEFINITION 3.12. We define the *Cheeger constant* of X as

$$h_m(X) := \inf\left\{\frac{P_m(D)}{\min\{\nu(D), \nu(X\setminus D)\}} \ : \ D \in \mathcal{B}, \ 0 < \nu(D) < 1\right\},$$

or, equivalently,

$$h_m(X) = \inf\left\{\frac{P_m(D)}{\nu(D)} \ : \ D \in \mathcal{B}, \ 0 < \nu(D) \leqslant \frac{1}{2}\right\}.$$

We have that $h_m(X) \leqslant 1$. If $h_m(X) > 0$, it is the best constant in the isoperimetric inequality

$$\lambda \min\{\nu(D), 1 - \nu(D)\} \leqslant P_m(D) \quad \text{for every } D \in \mathcal{B}.$$

Recall that in Section 3.1 we defined the m-Cheeger constant $h_1^m(\Omega)$ for sets $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \nu(X)$.

In this section, the m-Cheeger constant $h_m(X)$ is, instead, a global constant of the random walk space.

Observe that

$$h_m(X) \leqslant h_1^m(\Omega) \text{ for any } \Omega \in \mathcal{B} : 0 < \nu(\Omega) \leqslant 1/2;$$

and, if $h_m(X) = \frac{P_m(\Omega)}{\nu(\Omega)}$ for some $\Omega \in \mathcal{B}$ such that $0 < \nu(\Omega) \leqslant 1/2$, then

$$h_m(X) = h_1^m(\Omega)$$

and, moreover, Ω is m-calibrable.

Definition 3.13. Let $(X, \mathcal{B}, \nu)$ be a probability space and let $u : X \to \mathbb{R}$ be a measurable function. A real number μ is a *median* of u (with respect to ν) if

$$\nu(\{x \in X \ : \ u(x) < \mu\}) \leqslant \frac{1}{2} \quad \text{and} \quad \nu(\{x \in X \ : \ u(x) > \mu\}) \leqslant \frac{1}{2}.$$

We denote by $\text{med}_\nu(u)$ the set of medians of u.

Remark 3.14. It is easy to see that

$$\mu \in \text{med}_\nu(u) \ \Leftrightarrow \ -\nu(\{u = \mu\}) \leqslant \nu(\{u > \mu\}) - \nu(\{u < \mu\}) \leqslant \nu(\{u = \mu\}),$$

hence

$$0 \in \text{med}_\nu(u) \ \Leftrightarrow \ \exists \xi \in \text{sign}(u) \text{ such that } \int_X \xi(x) d\nu(x) = 0.$$

Moreover,

$$\arg\min \left\{ \int_X |u - c| d\nu \ : \ c \in \mathbb{R} \right\} = \text{med}_\nu(u). \tag{3.3}$$

The following variational characterization of the Cheeger constant generalizes the one obtained in [**45**, Szlam, Bresson] for the particular case of finite graphs.

THEOREM 3.15. *The following characterization of the Cheeger constant holds:*

$$h_m(X) = \inf \left\{ TV_m(u) : u \in L^1(X,\nu), \|u\|_{L^1(X,\nu)} = 1 \ \& \ 0 \in \mathrm{med}_\nu(u) \right\}.$$

PROOF. If $D \in \mathcal{B}$ satisfies $0 < \nu(D) \leqslant \frac{1}{2}$, then $0 \in \mathrm{med}_\nu(\chi_D)$. Therefore,

$$\inf\{\ldots\} \leqslant TV_m\left(\frac{1}{\nu(D)}\chi_D\right) = \frac{1}{\nu(D)}P_m(D),$$

thus

$$\inf\{\ldots\} \leqslant h_m(X).$$

Take now $u \in L^1(X,\nu)$ such that $\|u\|_{L^1(X,\nu)} = 1$ and $0 \in \mathrm{med}_\nu(u)$. Since $0 \in \mathrm{med}_\nu(u)$, by the coarea formula, we obtain that

$$\begin{aligned} TV_m(u) &= \int_{-\infty}^{+\infty} P_m(E_t(u))\,dt = \int_0^{+\infty} P_m(E_t(u))\,dt + \int_{-\infty}^0 P_m(X \setminus E_t(u))\,dt \\ &\geqslant h_m(X)\int_0^{+\infty} \nu(E_t(u))\,dt + h_m(X)\int_{-\infty}^0 \nu(X\setminus E_t(u))\,dt \\ &= h_m(X)\left(\int_X u^+(x)d\nu(x) + \int_X u^-(x)d\nu(x)\right) \\ &= h_m(X)\|u\|_{L^1(X,\nu)} = h_m(X). \end{aligned}$$

Therefore, taking the infimum over u, we get $\inf\{\ldots\} \geqslant h_m(X)$. $\square$

Following [**23**, Chung] and using Theorem 3.15, the Cheeger inequality also holds in our context.

THEOREM 3.16. *The following Cheeger inequality holds:*

$$\frac{(h_m(X))^2}{2} \leqslant \mathrm{gap}(-\Delta_m) \leqslant 2h_m(X).$$

COROLLARY 3.17. *The following statements are equivalent:*

(i) $[X, \mathcal{B}, m, \nu]$ *satisfies a Poincaré inequality,*

(ii) $\mathrm{gap}(-\Delta_m) > 0$*,*

(iii) $h_m(X) > 0$*.*

PROOF OF THEOREM 3.16. Let $(f_n) \subset L^2(X,\nu)$ non null such that $\nu(f_n) = 0$ and

$$\lim_{n\to\infty} \frac{2\mathcal{H}_m(f_n)}{\|f_n\|_2^2} = \operatorname{gap}(-\Delta_m).$$

And let $\mu_n \in \operatorname{med}_\nu(f_n)$.

We begin with some calculations:

$$4\mathcal{H}_m(f_n) = \int_X \int_X (f_n(y) - \mu_n - (f_n(x) - \mu_n))^2 dm_x(y) d\nu(x)$$

$$= \int_X \int_X \left[(f_n(y) - \mu_n)^+ - (f_n(x) - \mu_n)^+ - ((f_n(y) - \mu_n)^- - (f_n(x) - \mu_n)^-)\right]^2$$

$$= \int_X \int_X \left((f_n(y) - \mu_n)^+ - (f_n(x) - \mu_n)^+\right)^2 + \int_X \int_X \left((f_n(y) - \mu_n)^- - (f_n(x) - \mu_n)^-\right)^2$$

$$-2 \int_X \int_X \left((f_n(y) - \mu_n)^+ - (f_n(x) - \mu_n)^+\right) \left((f_n(y) - \mu_n)^- - (f_n(x) - \mu_n)^-\right) .$$

Hence,

$4\mathcal{H}_m(f_n) \geqslant$

$$\int_X \int_X \left((f_n(y)-\mu_n)^+ - (f_n(x)-\mu_n)^+\right)^2 + \int_X \int_X \left((f_n(y)-\mu_n)^- - (f_n(x)-\mu_n)^-\right)^2.$$

On the other hand, since $\nu(f_n)=0$, we have

$$\int_X f_n^2(x)d\nu(x) \leqslant \int_X (f_n(x)-\mu_n)^2 d\nu(x) =$$

$$\int_X \left((f_n(x)-\mu_n)^+\right)^2 d\nu(x) + \int_X \left((f_n(x)-\mu_n)^-\right)^2 d\nu(x).$$

Therefore,

$$\frac{4\mathcal{H}_m(f_n)}{\|f_n\|_2^2} \geqslant \frac{\int_X \int_X \left((f_n(y)-\mu_n)^+ - (f_n(x)-\mu_n)^+\right)^2 dm_x(y)d\nu(x)}{\int_X \left((f_n(x)-\mu_n)^+\right)^2 d\nu(x) + \int_X \left((f_n(x)-\mu_n)^-\right)^2 d\nu(x)}$$
$$+\frac{\int_X \int_X \left((f_n(y)-\mu_n)^- - (f_n(x)-\mu_n)^-\right)^2 dm_x(y)d\nu(x)}{\int_X \left((f_n(x)-\mu_n)^+\right)^2 d\nu(x) + \int_X \left((f_n(x)-\mu_n)^-\right)^2 d\nu(x)}.$$

Now,

$$\frac{a+b}{c+d} \geqslant \min\left\{\frac{a}{c}, \frac{b}{d}\right\} \quad \text{for every } a, b, c, d \in \mathbb{R}^+,$$

and

$$\int_X \left((f_n(x)-\mu_n)^+\right)^2 d\nu(x) + \int_X \left((f_n(x)-\mu_n)^-\right)^2 d\nu(x) \geqslant \int_X f_n^2(x) d\nu(x) > 0.$$

So, we can assume, without loss of generality, that

$$\int_X \left((f_n(x)-\mu_n)^+\right)^2 d\nu(x) > 0,$$

and that

$$\frac{4\mathcal{H}_m(f_n)}{\|f_n\|_2^2} \geqslant \frac{\int_X\int_X \left((f_n(y)-\mu_n)^+ - (f_n(x)-\mu_n)^+\right)^2 dm_x(y) d\nu(x)}{\int_X \left((f_n(x)-\mu_n)^+\right)^2 d\nu(x)}.$$

By the Cauchy-Schwartz inequality, we have

$$\int_X \int_X \left| \left((f_n(y) - \mu_n)^+ \right)^2 - \left((f_n(x) - \mu_n)^+ \right)^2 \right| dm_x(y) d\nu(x)$$

$$= \int_X \int_X \left| (f_n(y) - \mu_n)^+ - (f_n(x) - \mu_n)^+ \right| \times$$

$$\times \left| (f_n(y) - \mu_n)^+ + (f_n(x) - \mu_n)^+ \right| dm_x(y) d\nu(x)$$

$$\leqslant \left(\int_X \int_X \left((f_n(y) - \mu_n)^+ - (f_n(x) - \mu_n)^+ \right)^2 dm_x(y) d\nu(x) \right)^{\frac{1}{2}} \times$$

$$\times \left(\int_X \int_X \left((f_n(y) - \mu_n)^+ + (f_n(x) - \mu_n)^+ \right)^2 dm_x(y) d\nu(x) \right)^{\frac{1}{2}} .$$

Now, by the invariance of ν with respect to m,

$$\int_X \int_X \left((f_n(y)-\mu_n)^+ + (f_n(x)-\mu_n)^+\right)^2 dm_x(y)d\nu(x)$$

$$\leqslant 4\int_X \left((f_n(x)-\mu_n)^+\right)^2 d\nu(x).$$

Thus,

$$\frac{4\mathcal{H}_m(f_n)}{\|f_n\|^2_{L^2(X,\nu)}} \geqslant \left(\frac{\frac{1}{2}\int_X\int_X \left|\left((f_n(y)-\mu_n)^+\right)^2 - \left((f_n(x)-\mu_n)^+\right)^2\right| dm_x(y)d\nu(x)}{\int_X \left((f_n(x)-\mu_n)^+\right)^2 d\nu(x)}\right)^2.$$

Then, since $0 \in \operatorname{med}_\nu\left(((f_n-\mu_n)^+)^2\right)$, by Theorem 3.15, we get

$$(h_m(X))^2 \leqslant \frac{4\mathcal{H}_m(f_n)}{\|f_n\|^2_{L^2(X,\nu)}},$$

and, consequently, taking limits as $n\to\infty$, we obtain

$$(h_m(X))^2 \leqslant 2\mathrm{gap}(-\Delta_m).$$

To prove the other inequality we assume that $\mathrm{gap}(-\Delta_m) > 0$. Then from the Poincaré's inequality (1.2) for $f = \chi_D$,

$$\mathrm{gap}(-\Delta_m)\nu(D)(1-\nu(D)) \leqslant P_m(D) \quad \text{for all } D \in \mathcal{B},\ 0 < \nu(D) < 1.$$

Then

$$\frac{\mathrm{gap}(-\Delta_m)}{2}\min\{\nu(D), 1-\nu(D)\} \leqslant P_m(D) \quad \text{for all } D \in \mathcal{B},\ 0 < \nu(D) < 1,$$

from this it follows that

$$\frac{\mathrm{gap}(-\Delta_m)}{2} \leqslant h_m(X).$$

□

—— Assume that ν is a probability measure. ——

3.4. More about m-eigenvalues of $-\Delta_1^m$.

PROPOSITION 3.18. *Let (λ, u) be an m-eigenpair of $-\Delta_1^m$. Then,*

(i) $\lambda = 0 \Leftrightarrow$ u is ν-a.e. a constant,
and $(0, 1)$ and $(0, -1)$ are m-eigenpairs of $-\Delta_1^m$.

(ii) $\lambda \neq 0 \Leftrightarrow$ there exists $\xi \in \mathrm{sign}(u)$ such that $\int_X \xi(x) d\nu(x) = 0$
$\Leftrightarrow$ $0 \in \mathrm{med}_\nu(u)$.

For finite graphs see Hein and Bühler in [**32**].

PROOF OF PROPOSITION 3.18. (i) If $\lambda = 0$, we have that $TV_m(u) = 0$. Then, u is ν-a.e. a constant. Since $\|u\|_{L^1(X,\nu)} = 1$, either $u = 1$ or $u = -1$ ν-a.e.
If u is ν-a.e. a constant then $TV_m(u) = 0$ and, $\lambda = 0$.

(ii) ($\Leftarrow$) If $\lambda = 0$, by (i), we have that $u = 1$ or $u = -1$ ν-a.e., and this is in contradiction with the existence of $\xi \in \mathrm{sign}(u)$ such that $\int_X \xi(x) d\nu(x) = 0$.

($\Rightarrow$) There exists $\xi \in \mathrm{sign}(u)$ and $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ antisymmetric satisfying

$$\lambda\xi(x) = -\int_X \int_X \mathbf{g}(x,y)\, dm_x(y).$$

Hence, by the reversibility of ν with respect to m, we have

$$\lambda \int_X \xi(x) d\nu(x) = -\int_X \int_X \mathbf{g}(x,y)\, dm_x(y) d\nu(x) = 0.$$

Therefore, since $\lambda \neq 0$,

$$\int_X \xi(x) d\nu(x) = 0.$$

□

If $\lambda \neq 0$ is an m-eigenvalue of $-\Delta_1^m$, then there exists an m-eigenvector u associated to λ such that $\nu(E_0(u)) > 0$.

PROPOSITION 3.19. *Let (λ, u) is an m-eigenpair of $-\Delta_1^m$ with $\lambda > 0$. Let $t \geqslant 0$. If $\nu(E_t(u)) > 0$, then*

$$\left(\lambda, \frac{1}{\nu(E_t(u))}\chi_{E_t(u)}\right)$$

is an m-eigenpair of $-\Delta_1^m$,

$$\lambda^m_{E_t(u)} = \lambda$$

and $E_t(u)$ is m-calibrable. Moreover, $\nu(E_t(u)) \leqslant \frac{1}{2}$.

PROOF. First observe that, since $0 \in \mathrm{med}_\nu(u)$, $\nu(E_0(u)) \leqslant \frac{1}{2}$, thus

$$\nu(E_t(u)) \leqslant \tfrac{1}{2} \text{ for every } t \geqslant 0.$$

Since (λ, u) is an m-eigenpair, there exist $\xi \in \mathrm{sign}(u)$ and $\mathbf{g}(x,y) \in \mathrm{sign}(u(y) - u(x))$ antisymmetric such that

$$-\int_X \mathbf{g}(x,y)\, dm_x(y) = \lambda \xi(x) \quad \text{for } \nu\text{-a.e. } x \in X.$$

Let $t \geqslant 0$ such that $\nu(E_t(u)) > 0$. Then,

$$\xi(x) = \begin{cases} 1 & \text{if } x \in E_t(u) \text{ (since } u(x) > t \geqslant 0 \text{ and } \xi \in \operatorname{sign}(u)), \\ \in [-1,1] & \text{if } x \in X \setminus E_t(u), \end{cases}$$

and, therefore, $\xi \in \operatorname{sign}(\chi_{E_t(u)})$. On the other hand,

$$\mathbf{g}(x,y) = \begin{cases} \in [-1,1] & \text{if } x, y \in E_t(u), \\ -1 & \text{if } x \in E_t(u),\ y \in X \setminus E_t(u) \text{ (since } u(x) > t \geqslant u(y)), \\ 1 & \text{if } x \in X \setminus E_t(u),\ y \in E_t(u) \text{ (since } u(y) > t \geqslant u(x)), \\ \in [-1,1] & \text{if } x, y \in X \setminus E_t(u), \end{cases}$$

and, consequently, $\mathbf{g}(x,y) \in \operatorname{sign}(\chi_{E_t(u)}(y) - \chi_{E_t(u)}(x))$. Therefore, we have that $\left(\lambda, \frac{1}{\nu(E_t(u))}\chi_{E_t(u)}\right)$ is an m-eigenpair of $-\Delta_1^m$.

By Theorem 3.11, we have that $E_t(u)$ is m-calibrable. $\square$

COROLLARY 3.20. *If $\lambda \neq 0$ is an m-eigenvalue of $-\Delta_1^m$ then*

$$h_m(X) \leqslant \lambda.$$

THEOREM 3.21. *Let $\Omega \in \mathcal{B}$ such that $0 < \nu(\Omega) \leqslant \frac{1}{2}$.*

(i) If Ω and $X \setminus \Omega$ are m-calibrable, then $\left(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega\right)$ is an m-eigenpair of $-\Delta_1^m$.

(ii) If $h_m(X) = \lambda_\Omega^m$, then Ω and $X \setminus \Omega$ are m-calibrable. Therefore, $\left(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega\right)$ is an m-eigenpair of $-\Delta_1^m$.

COROLLARY 3.22. *If $h_m(X)$ is a positive m-eigenvalue of $-\Delta_1^m$, then, for any eigenvector u associated to $h_m(X)$ and any $t \geqslant 0$ such that $\nu(E_t(u)) > 0$,*

$$\left(h_m(X), \frac{1}{\nu(E_t(u))}\chi_{E_t(u)}\right) \text{ is an } m\text{-eigenpair of } -\Delta_1^m,$$

$\nu(E_t(u)) \leqslant \frac{1}{2}$, and

$$h_m(X) = \lambda^m_{E_t(u)}.$$

Moreover, both $E_t(u)$ and $X \setminus E_t(u)$ are m-calibrable.

If $\Omega \in \mathcal{B}$, $\nu(\Omega) = \frac{1}{2}$ (thus $\lambda_\Omega^m = 2P_m(\Omega)$), we have that:

– Ω and $X \setminus \Omega$ are m-calibrable if, and only if, $\left(2P_m(\Omega), t\chi_\Omega - (2-t)\chi_{X\setminus\Omega}\right)$ is an m-eigenpair of $-\Delta_1^m$ for any $t \in [0,2]$.

– If $h_m(X) = 2P_m(\Omega)$ then $\left(2P_m(\Omega), t\chi_\Omega - (2-t)\chi_{X\setminus\Omega}\right)$ is an m-eigenpair of $-\Delta_1^m$ for all $t \in [0,2]$.

REMARK 3.23. As a consequence of the above results, for finite (finite vertices) connected weighted discrete graphs, we have that

$$h_m(X) \ \text{ is the first non-zero eigenvalue of } \ -\Delta_1^{m^G}$$

(this was already proved in [**19**, Chang], [**20**, Chang, Shao, Zhang], and [**32**, Hein, Bühler]). **To solve the optimal Cheeger cut problem** is enough to find an eigenvector associated u to $h_m(X)$:

$$\{E_0(u), X \setminus E_0(u)\} \text{ or } \{E_0(-u), X \setminus E_0(-u)\} \text{ is a Cheeger cut.}$$

In [39] you can find examples of a connected graph with infinite points but finite measure for which $gap(-\Delta_m) = 0 = h_m(X)$, or for which $h_m(X) > 0$ is not an m-eigenvalue.

CHAPTER 4

ROF-models in random walk spaces

4.1. Introduction

Let Ω be a rectangle in $\mathbb{R}^2$. Given a noisy/corrupted image $f : \Omega \to \mathbb{R}$ by an additive noise, the problem of removing the noise to get the "clean" image u is ill-posed.

To *clean* the image, Rudin, Osher and Fatemi [**44**] proposed its following (BV, L^2)-decomposition:

$$f = u_\lambda + v_\lambda,$$

where

$$[u_\lambda, v_\lambda] = \underset{(u,v) \in BV(\Omega) \times L^2(\Omega)}{\arg\min} \left\{ \int_\Omega |Du| + \frac{\lambda}{2} \|v\|_2^2 \ : \ f = u + v \right\}.$$

wher λ is a kind of "scale parameter".

Following Meyer ([**40**]):

– The first component u_λ models the objects that are present in the image.

– The second component v_λ contains the textured parts and the noise.

By tweaking λ, we can select the level of detail desired in the reconstructed image.

If λ is too small then the regularization term $TV(u)$ is excessively penalized and the image is over-smoothed, resulting in a loss of information in the reconstructed image.

On the other hand, if λ is too large then the reconstructed image is under-regularized and noise is left in the reconstruction.

In [**44**], to solve the above ROF-problem, the gradient descent method was used, which required to solve numerically the parabolic problem

$$
(4.1) \qquad \begin{cases} u_t = \operatorname{div}\left(\frac{Du}{|Du|}\right) - \lambda(u - f) & \text{in } (0,\infty)\times\Omega, \\[2em] \frac{Du}{|Du|}\cdot\eta = 0 & \text{on } (0,\infty)\times\partial\Omega, \\[2em] u(0,x) = v_0(x) & \text{in } x\in\Omega. \end{cases}
$$

The denoised version of f is approached by the solution of (4.1) as t increases.

The concept of solution for which this problem is well-posed was given in [**6**, Andreu, Ballester, Caselles, Mazón] (see also [**7**, Andreu, Caselles, Mazón]).

The use of neighborhood filters has led to nonlocal models:

$$\min\left\{\int_{\Omega\times\Omega} J(x-y)|u(x)-u(y)|dxdy+\frac{\lambda}{2}\|u-f\|_2^2 \ : \ u\in L^2(\Omega)\right\}. \tag{4.2}$$

Since an image can be seen as a function on a weighted graph

where the pixels are taken as the vertices,

and the weights are related to the similarity between pixels,

one can also study the ROF-model in a weighted graph $G=(V(G),E(G))$:

$$\min_{u\in L^2(G,\nu_G)}\left\{\frac{1}{2}\sum_{x\in V(G)}\sum_{y\in V(G)}|u(y)-u(x)|w_{xy}+\frac{\lambda}{2}\sum_{x\in V(G)}|u(x)-f(x)|^2 d_x\right\}. \tag{4.3}$$

—— Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space with ν a probability measure. ——

4.2. The m-ROF model with L^2-fidelity term

Problems (4.2) and (4.3) are particular cases of the following m-ROF-model in $[X, \mathcal{B}, m, \nu]$:

$$\min_{u \in L^2(X,\nu)} \left\{ \frac{1}{2} \int_X \int_X |u(y) - u(x)| dm_x(y) d\nu(x) + \frac{\lambda}{2} \int_X |u(x) - f(x)|^2 d\nu(x) \right\},$$

or

$$\min \left\{ TV_m(u) + \frac{\lambda}{2} \|u - f\|^2_{L^2(X,\nu)} : \ u \in L^2(X,\nu) \right\}, \tag{4.4}$$

for $f \in L^2(X, \nu)$.

THEOREM 4.1. *For any $f \in L^2(X,\nu)$ and $\lambda > 0$, there exists a unique minimizer u_λ of problem (4.4). Moreover, u_λ is the unique solution of the problem*

$$\lambda(u - f) \in \Delta_1^m(u). \tag{4.5}$$

PROOF. Set

$$\mathcal{G}_m(u,f,\lambda) := TV_m(u) + \frac{\lambda}{2}\|u-f\|^2_{L^2(X,\nu)}, \quad u \in L^2(X,\nu).$$

Let $f \in L^2(X,\nu)$ and let $\{u_n\}_{n\in\mathbb{N}} \subset L^2(X,\nu)$ be a minimizing sequence of problem (4.4), i.e.,

$$\alpha := \inf\left\{\mathcal{G}_m(u,f,\lambda) \ : \ u \in L^2(X,\nu)\right\} = \lim_{n\to\infty} \mathcal{G}_m(u_n,f,\lambda).$$

Since

$$\|u_n\|^2_{L^2(X,\nu)} \leqslant 2\left(\|u_n - f\|^2_{L^2(X,\nu)} + \|f\|^2_{L^2(X,\nu)}\right) \leqslant 2\left(\frac{2}{\lambda}\mathcal{G}_m(u_n,f,\lambda) + \|f\|^2_{L^2(X,\nu)}\right),$$

we have that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(X,\nu)$ and we can assume that, up to a subsequence,

$$u_n \rightharpoonup u_\lambda \quad \text{weakly in } L^2(X,\nu).$$

Therefore, by lower semi-continuity with respect to the weak convergence in $L^2(X,\nu)$, we have that

$$\mathcal{G}_m(u_\lambda, f, \lambda) \leqslant \liminf_{n\to\infty} \mathcal{G}_m(u_n, f, \lambda) = \alpha,$$

hence u_λ is a minimizer of problem (4.4). The uniqueness of the minimizer follows from the strict convexity of $\|\cdot\|^2_{L^2(X,\nu)}$ and the convexity of TV_m.

Since u_λ is a minimizer of problem (4.4), we have that $0 \in \partial\mathcal{G}_m(u_\lambda, f, \lambda)$. Now, if $\Phi(u) := \frac{\lambda}{2}\|u-f\|^2_{L^2(X,\nu)}$, then, by [**14**, Brezis] we have that

$$\partial\mathcal{G}_m(u, f, \lambda) = \partial TV_m(u) + \partial\Phi(u),$$

thus

$$0 \in \partial\mathcal{G}_m(u_\lambda, f, \lambda) = \partial TV_m(u_\lambda) + \lambda(u_\lambda - f),$$

which yields (4.5). □

The m-ROF-model leads to the following (BV, L^2)-decomposition:

$$\begin{cases} f = u_\lambda + v_\lambda, \\ [u_\lambda, v_\lambda] = \displaystyle\operatorname*{arg\,min}_{(u,v)\in L^2(X,\nu)\times L^2(X,\nu)} \left\{ TV_m(u) + \frac{\lambda}{2}\|v\|^2_{L^2(X,\nu)} \ : \ f = u + v \right\}. \end{cases}$$

We have that

$$v_\lambda = \operatorname{div}_m(\mathbf{z}), \ \mathbf{z} \in L^\infty(X \times X, \nu \otimes m_x),$$

$$\|v_\lambda\|_{m,*} \leqslant \frac{1}{\lambda} \text{ and}$$

$$\lambda \int_X v_\lambda u_\lambda d\nu = TV_m(u_\lambda),$$

where

$$\|g\|_{m,*} := \inf \left\{ \|\mathbf{z}\|_{L^\infty(X\times X,\nu\otimes m_x)} \ : \ g = \operatorname{div}_m(\mathbf{z}) \right\}, \ g \in L^2(X, \nu).$$

Multiscale decomposition:

We have that

$$\|f\|_{m,*} \leqslant \frac{1}{\lambda} \quad \Leftrightarrow \quad u_\lambda = 0.$$

So, for continuing the *cleaning* by using v_λ as a image to clean, we need to use

$$\lambda_2 > \frac{1}{\|v_\lambda\|_{m,*}}.$$

PROPOSITION 4.2. *Let $f \in L^2(X,\nu)$. If $u_\lambda \in L^2(X,\nu)$ is the unique minimizer of problem* (4.4) *then*

$$\int_X u_\lambda(x)\,d\nu(x) = \int_X f(x)\,d\nu(x).$$

PROPOSITION 4.3 (Maximum Principle). *Let $f_1,\ f_2 \in L^2(X,\nu)$. If $[u_{i,\lambda}, v_{i,\lambda}]$ is the (BV, L^2)-decomposition of f_i, $i = 1, 2$, then*

$$\|(u_{1,\lambda} - u_{2,\lambda})^+\|_{L^2(X,\nu)} \leqslant \|(f_1 - f_2)^+\|_{L^2(X,\nu)}.$$

In particular, for $c, C \in \mathbb{R}$, if $c \leqslant f \leqslant C$ ν-a.e., and $[u_\lambda, v_\lambda]$ is the (BV, L^2)-decomposition of f, then

$$c \leqslant u_\lambda \leqslant C \quad \nu\text{-a.e.}$$

4.2.1. The Gradient Descent Method. Consider the following Cauchy problem:

$$\begin{cases} v_t \in \Delta_1^m v(t) - \lambda(v(t) - f) & \text{in } (0,T) \times X \\ v(0,x) = v_0(x) & \text{in } x \in X, \end{cases}$$

with v_0 satisfying $\int_\Omega v_0 = \int_\Omega f$.

THEOREM 4.4. *Such problem has a solution, that preserves the mass, and*

$$(4.6) \qquad \|v(t) - u_\lambda\|_{L^2(X,\nu)} \leqslant \|v_0 - u_\lambda\|_{L^2(X,\nu)}\, e^{-\lambda t} \quad \textit{for all } t \geqslant 0,$$

where u_λ is the unique minimizer of problem (4.4) *for such f.*

PROOF. We have

$$v_t + \lambda(v(t) - f) \in \Delta_1^m(v(t)),$$

and, by Theorem 4.1,

$$\lambda(u_\lambda - f) \in \Delta_1^m(u_\lambda).$$

Now, since $-\Delta_1^m$ is a monotone operator in $L^2(X,\nu)$, we get

$$\int_X (v(t) - u_\lambda)(-v_t - \lambda(v(t) - f) - (-\lambda(u_\lambda - f))d\nu \geqslant 0,$$

from where it follows that

$$\frac{1}{2}\frac{d}{dt}\int_X (v(t) - u_\lambda)^2 d\nu + \lambda \int_X (v(t) - u_\lambda)^2 d\nu \leqslant 0.$$

Then, integrating this ordinary differential inequality, we obtain (4.6). □

—— Let $[X, \mathcal{B}, m, \nu]$ be a reversible random walk space with ν a probability measure. ——

4.3. The m-ROF-model with L^1-fidelity term

In this section we will study the m-ROF-model with L^1-fidelity term, that is, given $f \in L^1(X, \nu)$ and $\lambda > 0$, we will study

$$\min \left\{ TV_m(u) + \lambda \int_X |u - f| d\nu \ : \ u \in L^1(X, \nu) \right\}.$$

See Alliney [**1, 2**], Chan, Esedoglu and Nikolova [**17, 18**] for the local problem.

The resulting (BV, L^1)-decomposition differs from the (BV, L^2)-one in several important aspects, for example, the (BV, L^1)-decomposition is contrast invariant ([**17**]), as opposed to the (BV, L^2)-decomposition.

We denote

$$\mathcal{E}_m(u,f,\lambda) := TV_m(u) + \lambda \int_X |u-f| d\nu, \quad u \in L^1(X,\nu).$$

And the set of minimizers of $\mathcal{E}_m(\cdot, f, \lambda)$ by $M(f,\lambda)$:

$$M(f,\lambda) := \left\{ u \in L^1(X,\nu) \ : \ \mathcal{E}_m(u,f,\lambda) = \inf_{\overline{u} \in L^1(X,\nu)} \mathcal{E}_m(\overline{u}, f, \lambda) \right\}.$$

This set can have several elements, it is convex and closed in $L^1(X,\nu)$.

In the local case, for every datum in L^1 a minimizer can be found via the direct method of the calculus of variations.

However, in our context, we do not have sufficient compactness properties in order to apply this method.

To prove that $M(f, \lambda) \neq \emptyset$ for every $f \in L^1(X, \nu)$ we study the geometric problem associated to the (BV, L^1)-decomposition (which is addressed in the next section).

PROPOSITION 4.5 (Maximum principle). *Let $f \in L^1(X,\nu)$, $\lambda > 0$ and $c, C \in \mathbb{R}$, and assume that $c \leqslant f \leqslant C$ ν-a.e. Then,*

$$\inf_{u \in L^1(X,\nu)} \mathcal{E}_m(u,f,\lambda) = \inf_{\substack{u \in L^1(X,\nu) \\ c \leqslant u \leqslant C}} \mathcal{E}_m(u,f,\lambda).$$

and, for any $u \in M(f,\lambda)$,

$$c \leqslant u \leqslant C \quad \nu\text{-a.e.}$$

THEOREM 4.6 (Euler-Lagrange equation). *Let $f \in L^2(X,\nu)$, $\lambda > 0$ and $u_\lambda \in L^2(X,\nu)$. Then, $u_\lambda \in M(f,\lambda)$ if, and only if, there exists $\xi \in sign(u_\lambda - f)$ such that*

$$\lambda\xi \in \Delta_1^m(u_\lambda).$$

EXERCISE 4.7 (Contrast invariance). Let $f \in L^2(X, \nu)$, $\lambda > 0$ and $T : \mathbb{R} \to \mathbb{R}$ a nondecreasing function. If $u_\lambda \in M(f, \lambda)$, then $T(u_\lambda) \in M(T(f), \lambda)$.

4.3.1. The Geometric Problem.

Given $F \in \mathcal{B}$ and $\lambda > 0$,

$$\mathcal{E}_m^G(A, F, \lambda) := \mathcal{E}_m(\chi_A, \chi_F, \lambda), \quad A \in \mathcal{B},$$

that is

$$\mathcal{E}_m^G(A, F, \lambda) = P_m(A) + \lambda\nu(A \bigtriangleup F).$$

THEOREM 4.8. *Let $u, f \in L^1(X, \nu)$ and $\lambda > 0$, then*

$$\mathcal{E}_m(u, f, \lambda) = \int_{-\infty}^{+\infty} \mathcal{E}_m^G(E_t(u), E_t(f), \lambda)\,dt.$$

For $\Omega \in \mathcal{B}$,

$$\mathcal{E}_m(u, \chi_\Omega, \lambda) = \int_0^1 \mathcal{E}_m^G(E_t(u), \Omega, \lambda)\,dt.$$

THEOREM 4.9. *Let $F \in \mathcal{B}$ be a non-ν-null set and $\lambda > 0$.*

(i) There exists a minimizer u_λ of $\mathcal{E}_m(\cdot, \chi_F, \lambda)$.

(ii) For a.e. $t \in]0,1[$, $E_t(u_\lambda)$ is a minimizer of $\mathcal{E}_m^G(\cdot, F, \lambda)$, and

$$\mathcal{E}_m(u_\lambda, \chi_F, \lambda) = \mathcal{E}_m^G(E_t(u_\lambda), F, \lambda).$$

PROOF. Since $\chi_F \in L^\infty(X,\nu)$, by the direct method of the calculus of variations, we have that there exists u_λ such that

$$\mathcal{E}_m(u_\lambda, \chi_F, \lambda) = \min_{u \in L^1(X,\nu)} \mathcal{E}_m(u, \chi_F, \lambda).$$

Indeed, by Proposition 4.5, there exists a minimizing sequence u_n with $0 \leqslant u_n \leqslant 1$, hence, bounded in $L^2(X,\nu)$. Then, by using Mazur's Lemma and the convexity of $\mathcal{E}_m(., \chi_F, \lambda)$, we get a minimizing sequence strongly convergent to some u_λ in $L^2(X,\nu)$. Now, by the lower semi-continuity of $\mathcal{E}_m(., \chi_F, \lambda)$ w.r.t. $L^1(X,\nu)$, we have that u_λ is a minimizer.

Now, by Theorem 4.8,

$$\int_0^1 \mathcal{E}_m^G(E_t(u_\lambda), F, \lambda)dt = \mathcal{E}_m(u_\lambda, \chi_F, \lambda) \leqslant \inf_{A \in \mathcal{B}} \mathcal{E}_m(\chi_A, \chi_F, \lambda) = \inf_{A \in \mathcal{B}} \mathcal{E}_m^G(A, F, \lambda),$$

hence, for a.e. $t \in]0,1[$,

$$\mathcal{E}_m^G(E_t(u_\lambda), F, \lambda) = \inf_{A \in \mathcal{B}} \mathcal{E}_m^G(A, F, \lambda),$$

which concludes the proof. $\square$

Adapting the ideas given in [**48**, Yin, Golfarb, Osher] for the local case, and thanks to the submodularity of the m-perimeter:

PROPOSITION 4.10. *Let $F_1, F_2 \in \mathcal{B}$, $F_1 \subseteq F_2$, and $\lambda > 0$. Suppose that $A_1, A_2 \in \mathcal{B}$ are minimizers of $\mathcal{E}_m^G(\cdot, F_1, \lambda)$ and $\mathcal{E}_m^G(\cdot, F_2, \lambda)$, respectively. Then, $A_1 \cap A_2$ and $A_1 \cup A_2$ minimize $\mathcal{E}_m^G(\cdot, F_1, \lambda)$ and $\mathcal{E}_m^G(\cdot, F_2, \lambda)$, respectively.*

THEOREM 4.11. *Given $f \in L^1(X, \nu)$ and $\lambda > 0$, there exists a function $u \in L^1(X, \nu)$ such that*

$$\mathcal{E}_m^G(E_t(u), E_t(f), \lambda) = \inf_{A \in \mathcal{B}} \mathcal{E}_m^G(A, E_t(f), \lambda) \quad \forall t \in \mathbb{R},$$

and it is a **minimizer** *of the variational problem*

$$\min_{u \in L^1(X,\nu)} \mathcal{E}_m(u, f, \lambda).$$

In the local case ([**25**, Duval, Aujol, Gousseau]) at points where the boundary of a minimizer E of the geometric problem for datum $F \subset \mathbb{R}^2$ and fidelity parameter λ does not coincide with the boundary of F, the mean curvature of ∂E is $\pm\lambda$.

There is a nonlocal counterpart of this fact where, the nonlocal character of the problem gives rise to a nontrivial extension. We state it for weighted graphs without loops:

Let $\lambda > 0$ and $F \in \mathcal{B}$ with $0 < \nu(F) < 1$, and let $E \in \mathcal{B}$ be a minimizer of $\mathcal{E}_m^G(\cdot, F, \lambda)$.
Then:

$$\max\left\{ \sup_{x \in F \cap E} \mathcal{H}_{\partial E}^{m^G}(x), \sup_{x \notin F \cup E} \left(-\mathcal{H}_{\partial E}^{m^G}(x) \right) \right\} \leqslant \lambda \leqslant \min\left\{ \inf_{x \in E \setminus F} \left(-\mathcal{H}_{\partial E}^{m^G}(x) \right), \inf_{x \in F \setminus E} \mathcal{H}_{\partial E}^{m^G}(x) \right\}.$$

4.3.2. Thresholding parameters.

In the local case it is well known ([**17**, Chan, Esedoglu]) that for $f = \chi_{B_r(0)}$ *the solution* u_λ of the problem is given by:

(i) $u_\lambda = \chi_{B_r(0)}$ if $\lambda \geqslant \frac{2}{r}$,

(ii) $u_\lambda = c\chi_{B_r(0)}$ with $0 \leqslant c \leqslant 1$ if $\lambda = \frac{2}{r}$,

(iii) $u_\lambda = 0$ if $0 < \lambda \leqslant \frac{2}{r}$, that is, it **suddenly vanishes**.

The thresholding property for a set in $\mathbb{R}^2$ implies (a) in Theorem 3.5, and both properties are equivalent for convex sets ([**25**, Duval, Aujol, Gousseau]).

We now see thresholding properties in the nonlocal case.

LEMMA 4.12. *For $\Omega \in \mathcal{B}$ and $\lambda > 0$,*

$$u_\lambda \in M(\chi_\Omega, \lambda) \Leftrightarrow 1 - u_\lambda \in M(\chi_{X\setminus\Omega}, \lambda).$$

LEMMA 4.13. *Let $f \in L^1(X, \nu)$ and $\lambda_0 > 0$.*

(i) If $f \in M(f, \lambda_0)$ then

$$\{f\} = M(f, \lambda) \quad \forall \lambda > \lambda_0.$$

(ii) If $f \in L^2(X, \nu)$ and a constant $c \in M(f, \lambda_0)$ then $c \in \mathrm{med}_\nu(f)$,

$$\mathrm{med}_\nu(f) \subset M(f, \lambda_0),$$

and

$$\mathrm{med}_\nu(f) = M(f, \lambda) \quad \forall 0 < \lambda < \lambda_0.$$

(iii) Let $\lambda_0 < \lambda_1$. If $u \in M(f, \lambda_0) \cap M(f, \lambda_1)$ then $u \in M(f, \lambda)$ for every $\lambda_0 \leqslant \lambda \leqslant \lambda_1$.

PROOF. (i): Take $\lambda > \lambda_0$, then, for any $u \in L^1(X,\nu)$ such that $\nu(\{u \neq f\}) > 0$, we have

$$\mathcal{E}_m(f,f,\lambda) = TV_m(f) = \mathcal{E}_m(f,f,\lambda_0) \leqslant \mathcal{E}_m(u,f,\lambda_0) < \mathcal{E}_m(u,f,\lambda).$$

(ii): Since $c \in M(f,\lambda_0)$ we have that, by Theorem 4.6, there exists $\xi \in \operatorname{sign}(c - f)$ and $\mathbf{g} \in L^\infty(X \times X, \nu \otimes m_x)$ antisymmetric satisfying

$$\int_X \mathbf{g}(x,y)\, dm_x(y) = \lambda_0 \xi(x) \quad \text{for } \nu\text{-a.e } x \in X \text{ and}$$

$$\mathbf{g}(x,y) \in \operatorname{sign}(0) \quad \text{for } (\nu \otimes m_x)\text{-a.e. } (x,y) \in X \times X.$$

Then,

$$\int_X \xi d\nu(x) = \frac{1}{\lambda_0} \int_X \int_X \mathbf{g}(x,y)\, dm_x(y) d\nu(x) = 0,$$

so that $0 \in \operatorname{med}_\nu(c - f)$, which is equivalent to $c \in \operatorname{med}_\nu(f)$. Now, for $\lambda < \lambda_0$, taking $g_\lambda(x,y) = \frac{\lambda}{\lambda_0} g(x,y)$ we obtain that

$$c \in M(f,\lambda).$$

Furthermore, by (3.3), for any other $m \in \mathrm{med}_\nu(f)$ and any $\lambda > 0$,

$$\mathcal{E}(c, f, \lambda_0) = \lambda \int_X |c - f| d\nu = \lambda \int_X |m - f| d\nu = \mathcal{E}(m, f, \lambda),$$

so that

$$\mathrm{med}_\nu(f) \subset M(f, \lambda_0), \quad \forall 0 < \lambda \leqslant \lambda_0.$$

Now, let $m \in \mathrm{med}_\nu(f)$, for any constant function $k \notin \mathrm{med}_\nu(f)$, by (3.3) we have that

$$\int_X |k - f| d\nu > \int_X |m - f| d\nu$$

so $k \notin M(f, \lambda)$ for every $\lambda > 0$.

Suppose then that there exists some nonconstant function u, such that $u \in M(f,\lambda)$ for $0 < \lambda < \lambda_0$. Since ν is ergodic with respect to m we have that $TV_m(u) > 0$, thus $\mathcal{E}(u,f,\lambda) \leqslant \mathcal{E}(m,f,\lambda)$ implies that $\int_X |u-f|d\nu < \int_X |m-f|$, and therefore

$$\begin{aligned}\mathcal{E}(u,f,\lambda_0) &= \mathcal{E}(u,f,\lambda) + (\lambda_0-\lambda)\int_X |u-f|d\nu \\ &< \mathcal{E}(m,f,\lambda) + (\lambda_0-\lambda)\int_X |m-f|d\nu = \mathcal{E}(m,f,\lambda_0)\end{aligned}$$

which is a contradiction. Consequently,

$$\mathrm{med}_\nu(f) = M(f,\lambda) \quad \forall 0 < \lambda < \lambda_0.$$

(iii) follows easily. □

PROPOSITION 4.14. *Assume that $[X, \mathcal{B}, m, \nu]$ is m-connected. Let (λ_0, u_0) be an m-eigenpair of $-\Delta_1^m$ with $\lambda_0 > 0$. Then,*

$0 \in med_\nu(u_0)$ and

$$\begin{cases} \{u_0\} = M(u_0, \lambda) & \text{if } \lambda > \lambda_0, \\ \{cu_0 \, : \, 0 \leqslant c \leqslant 1\} \cup med_\nu(u_0) \subset M(u_0, \lambda_0) & \\ med_\nu(u_0) = M(u_0, \lambda) & \text{if } 0 < \lambda < \lambda_0. \end{cases}$$

PROOF. Since (λ_0, u_0) is an m-eigenpair of $-\Delta_1^m$ with $\lambda_0 > 0$, we have that $0 \in \text{med}_\nu(u_0)$. Furthermore, by the definition of m-eigenpair, we have that

$$\exists \xi_0 \in \text{sign}(u_0) \text{ such that } -\lambda_0 \xi_0 \in \Delta_1^m(u_0).$$

Hence, for $0 < c \leqslant 1$, $\xi := -\xi_0 \in \text{sign}(cu_0 - u_0)$ and $\lambda_0 \xi \in \Delta_1^m(u_0) = \Delta_1^m(cu_0)$, which implies that $cu_0 \in M(u_0, \lambda_0)$. Moreover, since $TV_m(u_0) = \lambda_0$ (see Remark 3.9) and $\|u_0\|_{L^1(X,\nu)} = 1$, we have that

$$\mathcal{E}(u_0, u_0, \lambda_0) = \lambda_0 = \mathcal{E}(0, u_0, \lambda_0).$$

Consequently, by Lemma 4.13, we get the rest of the thesis. $\square$

PROPOSITION 4.15. *Assume that $[X, \mathcal{B}, m, \nu]$ is m-connected. Let $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < 1$. The following statements are equivalent:*

(i) $\chi_\Omega \in M(\chi_\Omega, \lambda_\Omega^m)$,

(ii) $\left(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega\right)$ *is an m-eigenpair (hence Ω is calibrable),*

(iii) the following thresholding property holds:

$$\begin{cases} \chi_\Omega \in M(\chi_\Omega, \lambda) & \forall\, \lambda \geqslant \lambda_\Omega^m, \\ 0 \in M(\chi_\Omega, \lambda) & \forall\, 0 < \lambda \leqslant \lambda_\Omega^m, \end{cases}$$

(iv) there exists a thresholding parameter $\lambda^* > 0$ such that

$$\begin{cases} \chi_\Omega \in M(\chi_\Omega, \lambda) & \forall\, \lambda > \lambda^*, \\ 0 \in M(\chi_\Omega, \lambda) & \forall\, 0 < \lambda < \lambda^*, \end{cases}$$

($\Rightarrow \lambda^* = \lambda_\Omega^m$).

For $\Omega \in B$, set

$$\lambda_*(\Omega) := \|\chi_\Omega - m_{(.)}(\Omega)\|_{L^\infty(X,\nu)}.$$

We have that

$$\max\{\lambda_\Omega^m, \lambda_{X\setminus\Omega}^m\} \leqslant \lambda_*(\Omega).$$

THEOREM 4.16. *Assume that $[X, \mathcal{B}, m, \nu]$ is m-connected and let $\Omega \in \mathcal{B}$. There exists $\lambda(\Omega) \in \mathbb{R}$,*

$$max\{\lambda_\Omega^m, \lambda_{X\setminus\Omega}^m\} \leqslant \lambda(\Omega) \leqslant \lambda_*(\Omega)$$

such that

$$\begin{cases} \{\chi_\Omega\} = M(\chi_\Omega, \lambda) & \text{if } \lambda > \lambda(\Omega), \\ \chi_\Omega \in M(\chi_\Omega, \lambda(\Omega)), & \\ \chi_\Omega \notin M(\chi_\Omega, \lambda) & \text{if } 0 < \lambda < \lambda(\Omega). \end{cases}$$

Furthermore,

$$\lambda(\Omega) = \lambda_\Omega^m \text{ if, and only if, } \left(\lambda_\Omega^m, \frac{1}{\nu(\Omega)}\chi_\Omega\right) \text{ is an } m\text{-eigenpair,}$$

and

$$\lambda(\Omega) = \lambda_{X\setminus\Omega}^m \text{ if, and only if, } \left(\lambda_{X\setminus\Omega}^m, \frac{1}{\nu(X\setminus\Omega)}\chi_{X\setminus\Omega}\right) \text{ is an } m\text{-eigenpair.}$$

We have:

$$\lambda(\Omega) = \sup\left\{\frac{P_m(\Omega) - P_m(E)}{\nu(\Omega \bigtriangleup E)} \,:\, E \in \mathcal{B},\ \nu(\Omega \bigtriangleup E) > 0\right\}.$$

We now provide some results regarding a thresholding parameter under which the set of minimizers are constant functions.

PROPOSITION 4.17. *Let* $\Omega \in \mathcal{B}$.

(i) If there exists $\lambda > 0$ *such that* $0 \in M(\chi_\Omega, \lambda)$, *then there exists* $\lambda^0(\Omega)$ *with* $0 < \lambda^0(\Omega) \leqslant h_1^m(\Omega)$ *such that*

$$\begin{cases} 0 \notin M(\chi_\Omega, \lambda) & \text{if } \lambda > \lambda^0(\Omega), \\ 0 \in M(\chi_\Omega, \lambda^0(\Omega)), \end{cases}$$

($\Rightarrow$ $med_\nu(\chi_\Omega) = M(\chi_\Omega, \lambda)$ *for* $0 < \lambda < \lambda^0(\Omega)$).

(ii) If there exists $\lambda > 0$ *such that* $1 \in M(\chi_\Omega, \lambda)$, *then there exists* $\lambda^1(\Omega)$ *with* $0 < \lambda^1(\Omega) \leqslant h_1^m(X \setminus \Omega)$ *such that*

$$\begin{cases} 1 \notin M(\chi_\Omega, \lambda) & \text{if } \lambda > \lambda^1(\Omega), \\ 1 \in M(\chi_\Omega, \lambda^1(\Omega)), \end{cases}$$

($\Rightarrow$ $med_\nu(\chi_\Omega) = M(\chi_\Omega, \lambda)$ *for* $0 < \lambda < \lambda^1(\Omega)$).

We can set $\lambda^0(\Omega) = 0$ if there is no $\lambda > 0$ such that $0 \in M(\chi_\Omega, \lambda)$, and $\lambda^1(\Omega) = 0$ if there is no $\lambda > 0$ such that $1 \in M(\chi_\Omega, \lambda)$.

$$\lambda^0(\Omega) = \inf \left\{ \frac{P_m(E)}{\nu(\Omega) - \nu(\Omega \triangle E)} \ : E \in \mathcal{B},\ \nu(\Omega \triangle E) < \nu(\Omega) \right\}.$$

PROPOSITION 4.18. *Let $0 < \nu(\Omega) < 1$. If $\lambda^0(\Omega) \geqslant \lambda_\Omega^m$ then Ω is m-calibrable.*

The following example proves that when the image is the characteristic function of a set Ω, the minimizer does not have be the characteristic function of a set contained in Ω.

We will observe how the solutions remain the same between certain parameters and make sudden transitions at certain values. In particular, we see how a set may suddenly vanish.

In the continuous setting, when Ω is a bounded convex domain, for almost all $\lambda \geqslant 0$ there is a unique minimizer which, moreover, is the characteristic function of a set contained in Ω (see [**17**, Chan, Esedoglu]).

EXAMPLE 4.19. Image: $\chi_{\{1,2\}}$.

In red the minimizers u_λ for the indicated parameter λ.

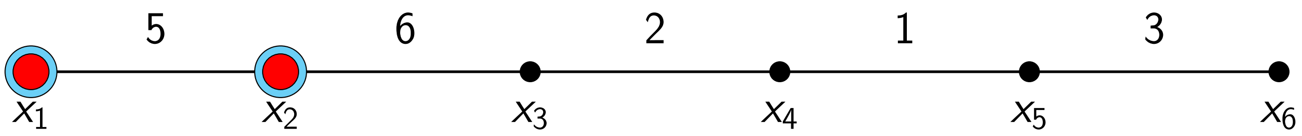


FIGURE 1. $\lambda > \frac{1}{2} = \lambda(\Omega)$, $M(\chi_{\{1,2\}}, \frac{1}{2}) = \{\chi_{\{1,2\}} + c\chi_{\{3\}} \ : \ c \in [0,1]\}$.

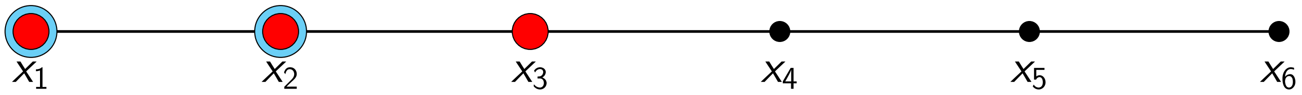


FIGURE 2. $\frac{1}{3} < \lambda < \frac{1}{2}$, $M(\chi_{\{1,2\}}, \frac{1}{3}) = \{\chi_{\{1,2,3\}} + c\chi_{\{4\}} \ : \ c \in [0,1]\}$.

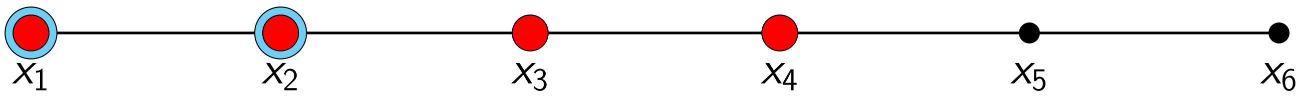


FIGURE 3. $\frac{1}{5} < \lambda < \frac{1}{3}$, $M(\chi_{\{1,2\}}, \frac{1}{5}) = \{c\chi_{\{1,2,3,4\}} \ : \ c \in [0,1]\}$.

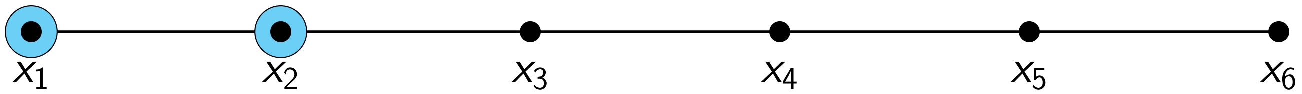
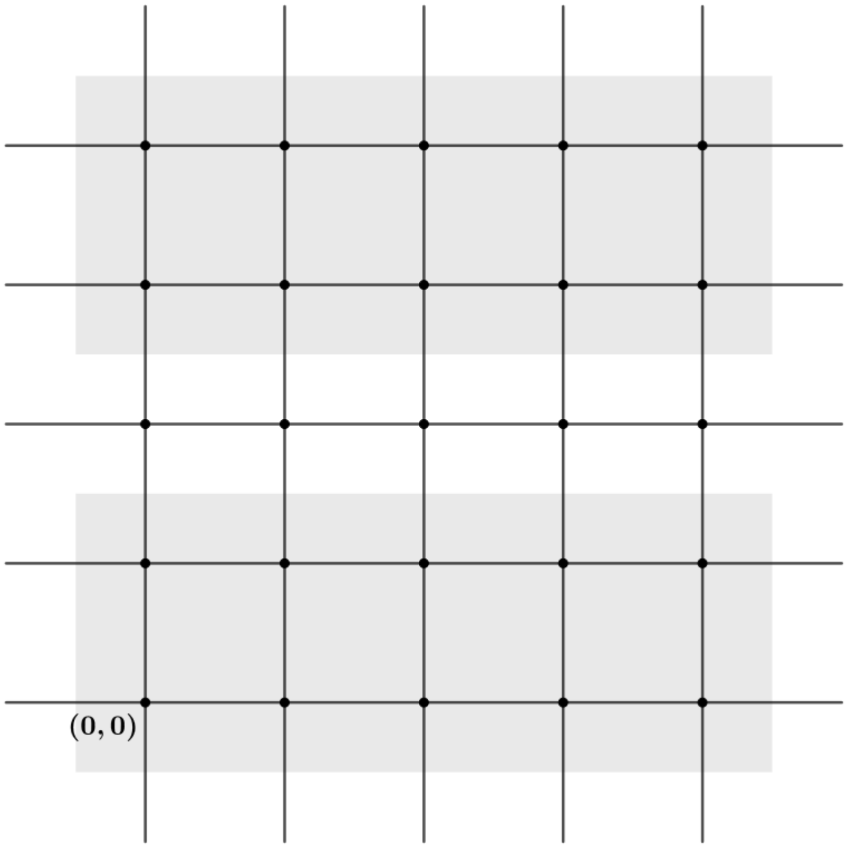
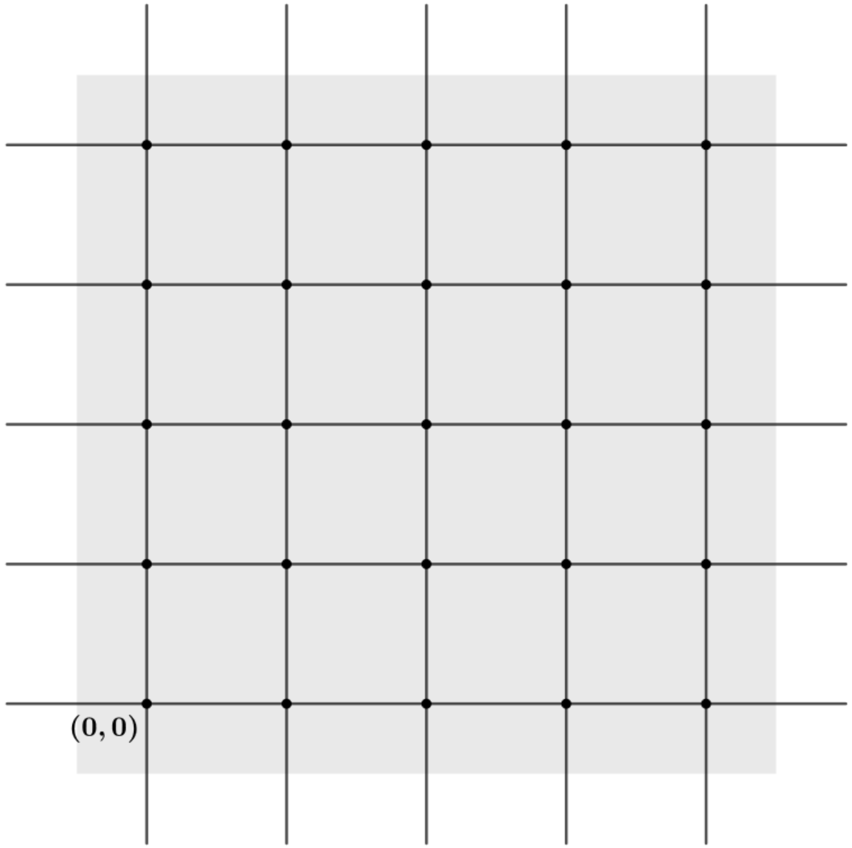


FIGURE 4. $\lambda < \frac{1}{5} = \lambda^0(\Omega)$.

EXAMPLE 4.20.



(A) Ω is the set formed by the points inside the shaded region.



(B) The minimizer, E, for $\frac{1}{3} < \lambda < \frac{2}{5}$ is the set formed by the points inside the shaded region.

FIGURE 5. The point $(0, 0)$ is labelled in the graphs, and the adjacent points are represented by dots.

4.3.3. The Gradient Descent Method. In order to apply this method one needs to solve the Cauchy problem

$$\begin{cases} v_t \in \Delta_1^m v(t) - \lambda \mathrm{sign}(v(t) - f) & \text{in } (0,T) \times X \\ v(0,x) = v_0(x) & \text{in } X, \end{cases} \tag{4.7}$$

that can be rewritten as the following abstract Cauchy problem in $L^2(X,\nu)$

$$v'(t) + \partial \mathcal{E}_m(u, f, \lambda)(v(t)) \ni 0, \quad v(0) = v_0. \tag{4.8}$$

Let f be in $L^1(X,\nu)$. Since $\mathcal{E}_m(\cdot,f,\lambda)$ is convex and lower semi-continuous, by the theory of maximal monotone operators ([**14**]), we have that, for any initial data $v_0 \in L^2(X,\nu)$, problem (4.8) has a unique strong solution.

THEOREM 4.21. *For every $v_0 \in L^2(X,\nu)$ there exists a unique strong solution of the Cauchy problem* (4.7) *in $(0,T)$ for any $T>0$. Moreover, we have the following contraction principle in any $L^q(X,\nu)$–space, $1 \leqslant q \leqslant \infty$:*

$$\|v(t)-w(t)\|_{L^q(X,\nu)} \leqslant \|v_0-w_0\|_{L^q(X,\nu)} \quad \forall\, 0<t<T,$$

for any pair of solutions v, w of problem (4.7) *with initial datum v_0 and w_0, respectively.*

THEOREM 4.22. *Assume that* $f \in L^1(X,\nu)$. *Let* $v_0 \in L^2(X,\nu)$ *and* $v(t) := T_\lambda(t)v_0$. *If the* ω*-limit set*

$$\omega(v_0) := \{w \in L^2(X,\nu) \ : \ \exists t_n \to +\infty \ s.t. \lim_{n\to\infty} v(t_n) = w\}$$

is non-empty, then there exists $u^* \in M(f,\lambda)$ *such that*

$$\lim_{t\to\infty} v(t) = u^* \quad \textit{in } L^2(X,\nu).$$

Proving that the ω-limit set $\omega(v_0)$ is non-empty is not an easy task here. For example, one could try to proceed with the usual method of proving that the resolvent is compact, but this requires the use of regularity results which are difficult to obtain in our context due to the non-locality of the problem. Nonetheless, in finite graphs it is trivially true that the ω-limit set is non-empty. Consequently, we have the following result.

COROLLARY 4.23. *Let* $[V(G), d_G, m^G, \nu_G]$ *be the metric random walk space associated to a locally finite weighted discrete graph* $G = (V(G), E(G))$. *Suppose that* ν_G *is a probability measure. Then, for every* $v_0 \in L^2(V(G), \nu_G)$ *and for* $v(t) := T_\lambda(t)v_0$, *there exists* $u^* \in M(f, \lambda)$ *such that*

$$\lim_{t\to\infty} v(t) = u^* \quad \text{in } L^2(V(G), \nu_G).$$

Bibliography

[1] S. Alliney, *Digital Filters as Absolute Norm Regularizers*. IEEE Trans. on Signal Processing, **40** (1992), 1548–1562.

[2] S. Alliney, *A Property of the Minimum Vectors of a Regularizing Functional Defined by Means of the Absolute Norm*. IEEE Trans. on Signal Processing, **45** (1997), 913–917.

[3] N. Alon and V.D. Milman, λ_1*, Isoperimetric inequalities for graphs, and superconcentrators*. J. Combin. Theory Ser. B **38**, (1985) 73–88.

[4] F. Alter, V. Caselles and A. Chambolle, *A characterization of convex calibrable sets in* $\mathbb{R}^N$*.* Math. Ann. **332** (2005), 329–366.

[5] L. Ambrosio, N. Fusco and D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems. Oxford Mathematical Monographs, 2000.

[6] F. Andreu, C. Ballester, V. Caselles and J. M. Mazón, *Minimizing Total Variation Flow*. Differential and Integral Equations, **14** (2001), 321-360.

[7] . Andreu, V. Caselles, and J.M. Mazón, Parabolic Quasilinear Equations Minimizing Linear Growth Functionals. Progress in Mathematics, vol. 223, 2004. Birkhauser.

[8] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi and J. Toledo, Nonlocal Diffusion Problems. Mathematical Surveys and Monographs, vol. 165. AMS, 2010.

[9] D. Bakry, I. Gentil and M. Ledoux, Analysis and Geometry of Markov Diffusion Operators. Grundlehren der Mathematischen Wissenschafter, **348,** Springer, 2014.

[10] F. Bauer and J. Jost, *Bipartite and neighborhood graps and the spectrum of the normalized graph laplace operator*. Comm. in Analysis and geometry **21** (2013), 787–845.

[11] B. B. Bhattacharya and S. Mukherjee, *Exact and asymptotic results on coarse Ricci curvature of graphs*. Discrete Mathematics **338** (2015), 23–42.

[12] S. Bougleux, A. Elmoataz and M. Melkemi, *Discrete Regularization on Weighted Graphs for Image and Mesh Filtering* . Proceedings of the 1st International Conference on Scale Space and Variational Methods in Computer Vision (SSVM'07), Lecture Notes in Comput. Sci. 4485, Springer-Verlag, Berlin, 2007, pp. 128–139.

[13] S. Bougleux, A. Elmoataz, and M. Melkemi, *Local and Nonlocal Discrete Regularization on Weighted Graphs for Image and Mesh Processing*. International J. of Computer Vision, **84** (2009), 220–236.

[14] H. Brezis, Operateurs Maximaux Monotones. North Holland, Amsterdam, 1973.

[15] A. Buades, B. Coll, and J. M. Morel, *A review of image denoising algorithms, with a new one*. Multiscale Modeling and Simulation **4** (2005), 490-–530.

[16] T. Bühler and M. Hein, *Spectral Clustering based on the graph p-Laplacian.* In Proceedings of the 26th International Conference on Machine Learning, pp. 81–88. Omnipress, 2009.

[17] T. F. Chan and S. Esedoglu, *Aspects of the Total Variation Regularized L^1 Function Approximation*. SIAM Journal on Applied Mathematics **65** (2005), 1817–1837.

[18] T. F. Chan, S. Esedoglu, and M. Nikolova, *Algorithms for Finding Global Minimizers of Image Segmentation and Denoising Models*. SIAM Journal on Applied Mathematics **66** (2006),1632–1648.

[19] K. C. Chang, *Spectrum of the 1-Laplacian and Cheeger's Constant on Graphs*. Journal of Graph Theory **81** (2016), 167–207.

[20] K. C. Chang, S. Shao and D. Zhang, *The 1-Laplacian Cheeger Cut: Theory and Algorithms* Journal of Computational Mathematics **33** (2015), 443–467.

[21] K.C. Chang, S. Shao and D. Zhang, *Cheeger's cut, maxcut and the spectral theory of* 1*-Laplacian on graphs.* Sci. China Math. **60** (2017), 1963–1980.

[22] J. Cheeger, A lower bound for the smallest eigenvalue of the Laplacian. Problems in Analysis, (R. C. Gunning, ed.) Princeton Univ. Press (1970) 195–199.

[23] F. Chung, Spectral Graph Theory (CBMS Regional Conference Series in Mathematics, No. 92), American Mathematical Society, 1997.

[24] J. Dodziuk, *Difference equations, isoperimetric inequality and transience of certain random walks*. Trans. Amer. Math. Soc. **284** (1984), 787–794.

[25] V. Duval, J-F. Aujol and Y. Gousseau, *The TVL1 Model: a Geometric Point of View* . SIAM Journal on Multiscale Modeling and Simulation, **8** (2009), 154–189.

[26] A. Elmoataz, O. Lezoray and S. Bougleux, *Nonlocal discrete regularization on weighted graphs: a framework for image and manifold processing.* IEEE Trans. on Image Processing, **17** (2008), 1047–1060.

[27] A. Elmoataz, M. Toutain and D. Tenbrinck, *On the* p*-Laplacian and* ∞*-Laplacian on graphs with applications in image and data processing.* SIAM J. Imaging Sci. **8** (2015), 2412–2451.

[28] V. Fridman, B. Kawohl, *Isoperimetric estimates for the first eigenvalue of the* p*-Laplace operator and the Cheeger constant*. Comment. Math. Univ. Carolinae **44** (2003), 659–667.

[29] N. García-Trillos and D. Slepčev, *Continuum Limit of Total variation on Point Clouds*. Arch. Rat. Mech. Anal. Research **220** (2016), 193–241.

[30] N. García-Trillos, D. Slepčev, J von Brecht, T. Laurent and X. Bresson, *Consistency of Cheeger and Ratio Graph Cuts*. Journal of Machine Learning Research **17** (2016), 1–46.

[31] Y. van Gennip, N. Guillen, B. Osting and A, Bertozzi, *Mean Curvature, Threshold Dynamics, and Phase Field Theory on Finite Graphs* Milan J. Math. **82** (2014), 3–65.

[32] M. Hein and T. Bühler, *An Inverse Power Method for Nonlinear Eigenproblems with Applications in* 1*-Spectral Clustering and Sparse PCA*. Advances in Neural Informatio Proceessing Systems **23** (2010), 847–855.

[33] M. Hidane, O. Lézoray and A. Elmoataz, *Nonlinear Multilayered Representation of Graph-Signals*. Journal of Mathematical Imaging and Vision, **45** (2013),114–137

[34] M. Hidane, O. Lézoray, V.T. Ta and A. Elmoataz, *Nonlocal Multiscale Hierarchical Decomposition on Graphs*. Computer Vision ECCV 2010, 638–650. Lecture Notes in Computer Science, 6314. Springer, Berlin-Heidelberg, 2010.

[35] F. Lozes, A. Elmoataz and O. Lézoray, *Partial Difference Operators on Weighted Graphs for Image Processing on Surfaces and Point Clouds*. IEEE Trans. on Image Processing, **23** (2014), 3896–3909.

[36] U. von Luxburg, *A tutorial on spectral clustering.* Stat. Comput. **17** (2007), 395–416.

[37] J. M. Mazón, J. D. Rossi and J. Toledo, *Nonlocal Perimeter, Curvature and Minimal Surfaces for Measurable Sets*. J. Anal. Math. **138** (2019), no. 1, 235–279.

[38] J. M. Mazón, J. D. Rossi and J. Toledo, Nonlocal Perimeter, Curvature and Minimal Surfaces for Measurable Sets. Frontiers in Mathematics, Birkhäuser, 2019.

[39] J. M. Mazón, M. Solera and J. Toledo, Variational and Diffusion Problems in Random Walk Spaces, Progress in Nonlinear Differential Equations and Their Applications, vol. 103, Birkhäuser, 2023.

[40] Y. Meyer, scillating patterns in image processing and nonlinear evolution equations. University lecture Series, 22. American Mathematical Society, Providance, RI, 2001.

[41] Y. Ollivier, *Ricci curvature of Markov chains on metric spaces.* J. Funct. Anal. **256** (2009), 810–864.

[42] G. Polya and S. Szego, *Isoperimetric Inequalities in Mathematical Physics,* Annals of Math. Studies, no. 27, Princeton University Press, (1951).

[43] M.-K von Reness, K.-L. Sturm, *Transport Inequalities, Gradient Estimates, Entropy, and Ricci Curvature*, Communications on Pure and Applied Mathematics, Vol. LVIII, (2005), 0923–0940.

[44] L. Rudin, S. Osher and E. Fatemi, *Nonlinear Total Variation based Noise Removal Algoritms*. Physica D., **60** (1992), 259–268.

[45] A. Szlam and X. Bresson, *Total Variation and Cheeger Cuts*. Proceedings of the 27 th International Conference on Machine Learning, Haifa, Israel, 2010.

[46] C. Villani, Optimal transport. Old and new. Grundlehren der Mathematischen Wissenschaften, vol. 338, Springer-Verlag, Berlin, 2009.

[47] L. P. Yaroslavsky, Digital Picture Processing–An Introduction. Springer, Berlin, 1985.

[48] W. Yin, D. Golfarb and S. Osher, *The Total Variation Regularized L^1 Model for Multiscale Decomposition*. Multiscale Modeling and Simulation, **6** (2007), 190–211.