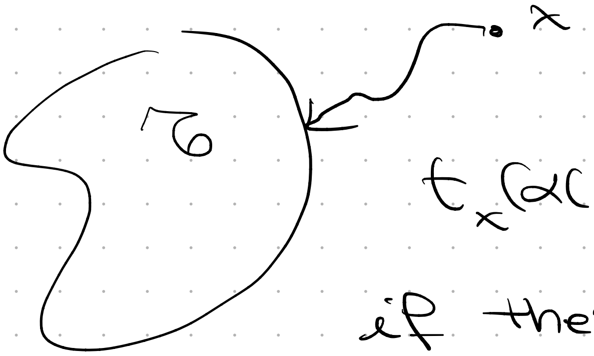
MINIMUM TIME PROBLEM

Given a dynamic $f : \mathbb{R}^n \times A \to \mathbb{R}^n$

$$(1)\ \begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) & s \geq 0 \\ y(0) = x \end{cases}$$

Given a **TARGET SET** $\mathcal{T} \neq \emptyset$ $\mathcal{T} \subseteq \mathbb{R}^n$



Let us define, **for a given α**

$$t_x(\alpha(\cdot)) = \inf\{t \in \mathbb{R}^+ \mid y_{x,\alpha}(t) \in \mathcal{T}\}$$

if there exists a t, $y_{x,\alpha}(t) \in \mathcal{T}$

otherwise $t_x(\alpha) = +\infty$

$$t_x(\alpha) = \begin{cases} \inf\{t \in \mathbb{R}^+ \mid y_{x,\alpha}(t) \in \mathcal{T}\} \\ +\infty & \text{if } y_{x,\alpha}(t) \notin \mathcal{T},\ t > 0 \end{cases}$$

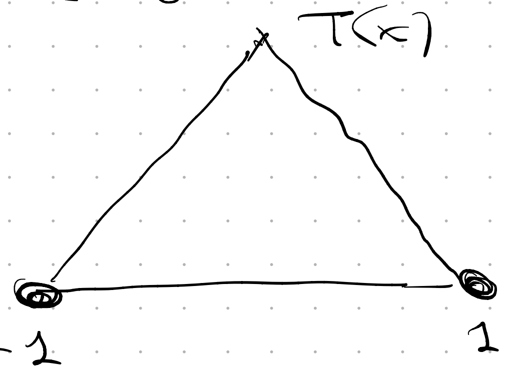
VALUE FUNCTION

$$\boxed{T(x) := \inf_{\alpha \in \mathcal{A}} t_x(\alpha(\cdot))}$$

the smallest over all the arrival times

Ex $n = 1$, $A = [-1, 1]$, $\mathcal{T} = \{-1\} \cup \{1\}$,

$f(x, a) = a$



$$\begin{cases} \dot{y}(s) = \alpha(s) \\ y(0) = x_0 \end{cases}$$

If $x_0 \in [0, 1]$, $\alpha^*(s) = 1$

$$1 = y^*(t) = x_0 + t \Rightarrow t = 1 - x_0$$

If $x_0 \in [-1,0]$, $\alpha^*(s) = -1$

$$-1 = y^*(t) = x_0 - t \Rightarrow t = 1 + x_0$$

Then for $x_0 \in [-1,1]$, $T(x) = \begin{cases} 1-x & x \in [0,1] \\ 1+x & x \in [-1,0] \end{cases}$

The minimum time problem is equivalent to minimize, with the same dynamic (D), the COST

$$J_x(\alpha(\cdot)) = \int_0^{t_x(\alpha)} 1\, ds = t_x(\alpha)$$

If the dynamic can remain fixed at the target

$$T(x) = \inf_{\alpha \in \mathcal{A}} \int_0^{+\infty} \mathbb{1}_{\mathcal{T}^c}(y_{x,\alpha}(s))\, ds =$$

$$= \inf_{\alpha \in \mathcal{A}} \Big[\int_0^{t_x(\alpha)} \mathbb{1}_{\mathcal{T}^c}(y_{x,\alpha}(s))\, ds + \underbrace{\int_{t_x(\alpha)}^{+\infty} \mathbb{1}_{\mathcal{T}^c}(y_{x,\alpha}(s))\, ds}_{0} \Big]$$

$\lambda = 0$

$$\ell(s, \alpha(s)) = \mathbb{1}_{\mathcal{T}^c}(y_{x,\alpha}(s)) = \begin{cases} 1 & y_{x,\alpha}(s) \notin \mathcal{T} \\ 0 & y_{x,\alpha}(s) \in \mathcal{T} \end{cases}$$

The related HJB equation: MINIMUM TIME EQ
For a general dynamic (D) in $\mathbb{R}^n$

$$\begin{cases} \max_{a \in A} [-\nabla v(x) \cdot f(x,a) - 1] = 0 & x \in \mathbb{R}^n \setminus \mathcal{T} \\ v(x) = 0 & x \in \mathcal{T} \end{cases}$$

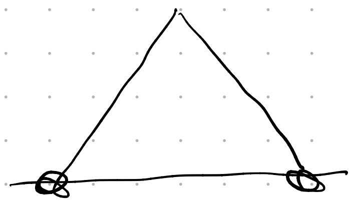
Going back to the example

the minimum time equation in $(-1,1)$

$$\begin{cases} \max_{a\in[-1,1]} [-v'(x)\cdot a - 1] = 0 & x\in(-1,1) \\ v(x) = 0 & |x| = 1 \end{cases} \qquad a^* = \frac{-v'(x)}{|v'(x)|}$$

Since $\max_{a\in[-1,1]} [-v'(x)a - 1] = \max_{a\in[-1,1]} [-v'(x)a] - 1 = |v'(x)| - 1$

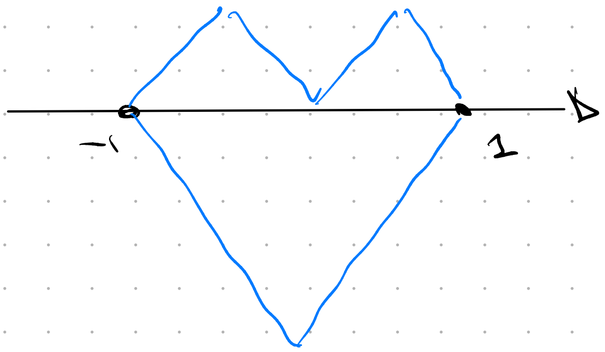
VALUE FUNCTION



the eq. reads

$$\begin{cases} |v'(x)| = 1 & x\in(-1,1) \qquad (1) \\ v(x) = 0 & |x| = 1 \end{cases}$$

$v(x)$ is lipschitz and differentiable almost everywhere



One possibility to select a unique solution is to regularize the problem adding a second order term $-\varepsilon v''$ and then pass to the limit $\varepsilon \to 0$.

$$(HJB)_\varepsilon \quad \begin{cases} -\varepsilon v''(x) + |v'(x)| = 1 & x \in (-1,1) \\ v(-1) = v(1) = 0 \end{cases}$$

- $\forall\, \varepsilon > 0$ there exist a unique $V_\varepsilon \in C^2(-1,1)$ solution to $(HJB)_\varepsilon$

Since

- $|V_\varepsilon(x) - V_\varepsilon(y)| \leq L|x-y|$ equicontinuous
- $|V_\varepsilon(x)| \leq M$ uniform bound

(L and M do not depend on ε)

We can pass to the limit (by Ascoli-Arzelà theorem)

$$\lim_{\varepsilon \to 0} V^\varepsilon(x) = V(x)$$

This defines the WEAK SOLUTION of the original problem (Kruskov '60)

VISCOSITY SOLUTION

The viscosity solution V is the uniform limit of V^ε where V^ε solve

$$\begin{cases} -\varepsilon \Delta V(x) + H(x, V, \nabla V) = 0 & x \in \Omega \subseteq \mathbb{R}^n \\ \text{b.c.} \end{cases}$$

for a given open domain $\Omega \subseteq \mathbb{R}^n$

$BUC(\Omega) = \{u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ bounded uniformly continuous$\}$

<u>DEF</u> Let $V \in BUC(\Omega)$ is a **VISCOSITY SOLUTION** of $\quad H(x, V(x), \nabla V(x)) = 0 \quad x \in \Omega$

if and only if for any $\phi \in C^1(\Omega)$ the

the following conditions hold

1) at every local maximum $x_0 \in \Omega$ for $V-\phi$

$$H(x_0, v(x_0), \nabla\phi(x_0)) \leq 0$$

(i.e. v is a VISCOSITY **<u>SUB</u> SOLUTION**)

2) at every local minimum $x_0 \in \Omega$ for $V-\phi$

$$H(x_0, v(x_0), \nabla\phi(x_0)) \geq 0$$

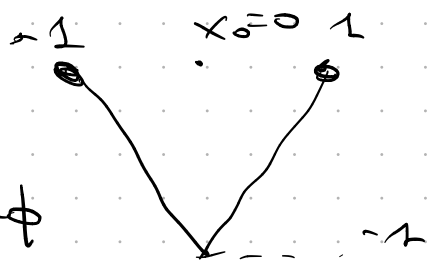
(i.e. v is a VISCOSITY **<u>SUPER</u> SOLUTION**)

<u>OSS</u> 1) We give a punctual definition on x_0 and we need to check for any test function

2) $\nabla\phi$ exist, but ∇v a priori does not

Going back to the <u>example</u>

Take any solution v which has a local minimum in x_0, choose as test function $\phi = \text{const}$



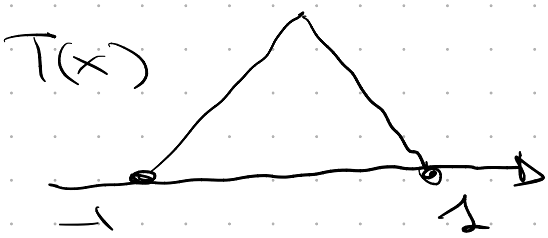
Clearly, x_0 is a minimum for $v-\phi$

$$(v-\phi)(x_0) \leq (v-\phi)(x)$$

$|\phi'(x_0)| \geq 1$ which is false since $\phi \equiv \text{const}$

Conclusion any almost everywhere solution having a local minimum cannot be a supersolution

$$\begin{cases} |u'(x)| = 1 & x \in (-1,1) \\ u(x) = 0 & |x| = 1 \end{cases}$$



So, T(x) the minimum time value function is the UNIQUE VISCOSITY SOLUTION

PROPERTIES of VISCOSITY SOLUTIONS

1. If $V(x)$ is a CLASSICAL solution C^1 then $V(x)$ is a VISCOSITY SOLUTION
2. If $V(x)$ is a REGULAR VISCOSITY solution then it is also CLASSICAL (i.e. it satisfies the equation point wise)
3. the viscosity solution is the MAXIMAL SUBSOLUTION, i.e

$$V \geq w \quad \text{for any } w \text{ sub sol}$$

CRUCIAL POINT: UNIQUENESS

Thm (Comparison Principle) Let

let $u, v \in BUC(\Omega)$ be respectively a sub and a supersolution s.t.

$$u(x) \leq v(x) \quad \text{for any } x \in \partial\Omega$$

then $\quad u(x) \leq v(x) \quad$ for any $x \in \Omega$.

VISCOSITY SOLUTION for STATIONARY FIRST ORDER EQUATION

Given $\Omega \subseteq \mathbb{R}^n$

$$\begin{cases} H(x, v(x), Dv(x)) = 0 & x \in \Omega \\ v(x) = g(x) & x \in \partial\Omega \end{cases} \qquad (HJ)$$

$H: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ is called the Hamiltonian and $g(x)$ represent a Dirichlet boundary condition

<u>DEF</u> $v \in BUC(\overline{\Omega})$ is a VISCOSITY SOLUTION to (HJ) if it is a viscosity solution on Ω and $v(x) = g(x)$ $x \in \partial\Omega$

Ω^c

<u>EX</u> PROBLEM wit EXIT TIME and general TARGET

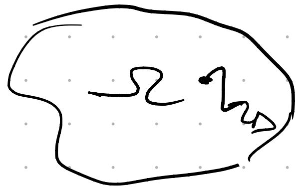
Consider Hamiltonian of the form

$$H(x, v, Dv) = \lambda v(x) + \sup_{a \in A} \{ -f(x,a) Dv(x) - l(x,a) \}$$

associated with the problem of minimizing the cost functional

$$J_\alpha(x) = \begin{cases} \displaystyle\int_0^{t_x(\alpha)} l(y_{x\alpha}(s), \alpha(s)) e^{-\lambda s} ds + e^{-\lambda t_x(\alpha)} g(y_{x\alpha}(t_x(\alpha))) & \text{if } t_x(\alpha) < \infty \\ \displaystyle\int_0^{\infty} l(y_{x,\alpha}(s), \alpha(s)) e^{-\lambda s} ds & \text{if } t_x(\alpha) = +\infty \end{cases}$$

$t_x(\alpha)$ is the FIRST EXIT TIME from the open set Ω, $\mathcal{T} = \Omega^c$



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The COMPARISON PRINCIPLE implies UNIQUENESS

Two viscosity solution v_1, v_2.
They are both super and sub solution by the comparison principle

$$v_1 \geq v_2 \,, \quad v_2 \geq v_1 \Rightarrow v_1 = v_2$$

The typical assumptions for EXISTENCE

(A)
- H uniformly continuous
- $H(x, v, \cdot)$ convex
- $H(x, \cdot, p)$ monotone

Th If (A) is satisfied, $u = g \in \partial\Omega$ and the comparison principle holds then (HJ) admits a unique viscosity solution

FINITE HORIZON CONTROL PROBLEM

Fix a terminal time $T>0$ and varying the starting point of the dynamic

$$(\tilde{D}) \quad \begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) \\ y(t) = x \end{cases}$$

t a general time

Let $y_{x,t}(s)$ solution to $(\tilde{D})$ and consider the cost, given $T>0$

$$J_{x,t}(\alpha(\cdot)) = \int_t^T \ell(y_{x,t}(s), \alpha(s))ds + \psi(y_{x,t}(T))$$

for all choices of (x,t)

<u>DEF</u> For $x \in \mathbb{R}^n$, $t \in [0,T]$ the VALUE FUNCTION

$$v(x,t) = \inf_{\alpha(\cdot)\in\mathcal{A}} J_{x,t}(\alpha(\cdot))$$

Notice that $v(x,T) = \psi(x)$ $x \in \mathbb{R}^n$

<u>TH</u> Assume that $v \in C^1(\mathbb{R}^n \times [0,T])$ then v satisfies

$$(HJB) \quad \begin{cases} -v_t(x,t) + \sup_{a\in A}\{-f(x,a)\nabla v(x,t) - \ell(x,a)\} = 0 \\ v(x,T) = \psi(x) \end{cases}$$

The hamiltonian

$$H(x, \nabla v) = \sup_{a \in A} \{ -f(x,a) \nabla v(x,t) - \ell(x,a) \}$$

and (HJB)
$$\begin{cases} -v_t + H(x, \nabla v(x,t)) = 0 \\ v(x,T) = \psi(x) \end{cases}$$

DYNAMIC PROGRAMMING PRINCIPLE: $\tau > 0$

$$v(x,t) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_t^{t+\tau} \ell(y_{x,t}(s), \alpha(s))\,ds + v(y_{x,t}(t+\tau), t+\tau) \right\}$$

FRONT PROPAGATION and MINIMUM TIME

EVOLUTIVE vs STATIONARY

The minimum time problem can be seen as an evolutive HJB by using the **LEVEL SET METHOD** great success for modeling and analysing of FRONT PROPAGATION

The front is represented by the level set of a function

$v : \mathbb{R}^n \times [0,T]$

$$\Gamma_t = \{x \in \mathbb{R}^n : v(x,t) = 0\}$$

The front propagates with normal velocity with **speed** $c(x)\mathbf{n}(x)$ where $c : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{n} : \mathbb{R}^n \to \mathbb{R}^n$ is the OUTWARD normal

Let us observe that

$$\mathbf{n}(x) = \frac{\nabla v(x,t)}{|\nabla v(x,t)|}$$

$$\begin{cases} \dot{y}(s) = c(y(s))\, n(y(s)) \\ y(t) = x \end{cases}$$

Then setting $\boxed{v(y(t),t)=0}$

$$0=\frac{d}{dt}v(y(t),t)=\nabla v(y(t),t)\cdot\dot{y}(t)+v_t(y(t),t)=$$

$$=\nabla v(y(t),t)\,c(y(t))\cdot n(y(t))+v_t(y(t),t)=$$

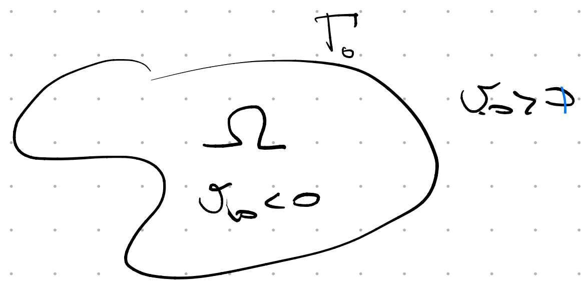
$$=\nabla v(y(t),t)\,c(y(t))\cdot\frac{\nabla v(y(t),t)}{|\nabla v(y(t),t)|}+v_t(y(t),t)$$

$$=c(y(t))\,|\nabla v(y(t),t)|+v_t(y(t),t)$$

Since $y(t)=x$

$$\begin{cases} v_t(x,t)+c(x)\,|\nabla v(x,t)|=0 \\ v(x,0)=v_0(x)\end{cases}$$

FRONT PROPAGATION PROBLEM



The front propagation problem corresponds to the MINIMUM TIME PROBLEM with $c(x)>0$

$$A=B(0,1),\quad f(x,a)=-c(x)a,\quad \mathcal{T}_0=\Omega$$

$$\ell(x,a)=0 \qquad \lambda=0$$

In fact

$$\max_{a\in B(0,1)}\{-c(x)\,a\cdot\nabla T(x)\}=+c(x)\,|\nabla T(x)|$$

$$a^*=-\frac{\nabla T(x)}{|\nabla T(x)|}$$

Then the minimum time problem

$$\begin{cases} c(x)\,|\nabla T(x)| = 1 & x \in \mathbb{R}^n \setminus \mathcal{T} \\ T(x) = 0 & x \in \mathcal{T} = \Omega \end{cases} \iff \begin{cases} v_t(x,t) + c(x)\,|\nabla v(x,t)| = 0 & \mathbb{R}^n \times (0,T) \\ v(x,0) = v_0(x) \end{cases}$$

REACHABLE SET at time t: minimum time to reach $\mathcal{T}$ from x <u>smaller than t</u>

$$R_t = \{x \in \mathbb{R}^n : T(x) < t\} = \{x : v(x,t) < 0\} = \Omega_t$$

$$\mathcal{T} = \{x : v_0(x) < 0\}$$